Lorentz Transformations

A Study of Some of Their Properties

by

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Introduction

This paper is a loose conversation about some selected topics which, in my opinion, are essential to understanding Lorentz transformations more profoundly. Having these thoughts written down at a central place not only avoids duplicate work in case I need the results again but also enables me to share them with others. To clarify matters, this discussion is not about the Lorentz transformations' physical content, it rather is about the algebraic group structure and the integration into a suitable mathematical framework. The resulting expectations on the reader are quite modest: If e.g. we need the spinor representation of the world vector with its transformations, we will shortly introduce the representation theory of Lorentz transformations beforehand. This text somehow builds upon my paper [3] about rotations, from which the mathematical approach is borrowed.

We assume the reader to be somewhat familiar with the physical background of Special Relativity because we will not cover that. Instead, we will focus on formal analysis and the choice of appropriate mathematical tools, which, in my opinion, is key to a deeper understanding of any physical phenomenon. Regarding rotations, the use of quaternions appears more natural and concise than fiddling with rotation matrices (aka fundamental representations of the SO(3) group) alone. Another handy mathematical instrument is the theory of Lie groups, which aside from conflating continuously parameterizable symmetry operations into a general formalism also provides for hands–on results like Rodrigues' rotation formula [3].

We will start with our list of chosen topics (or the table of contents). The chapters are more or less self–contained units, which means, a table of contents will allow you to conveniently pick out things you want to know more about. At least, if nothing sparks your interest, your loss of lifetime due to unnecessary reading is reduced to a minimum :-).

The translation from German to English is still ongoing, so please be prepared that –after a polite warning– the text switches from English to German. Over time, the location where this happens will move towards the end. Meanwhile, I have to apologize for this inconvenience...

Contents

Introduction . . . . **i**

1 The Foundation of Lorentz Transformations **1**

1.1 Isometric Requirement 1

1.2 Polar Decomposition 2

2 Derivation of the Lorentz Boost **4**

2.1 Solving the Equation System 4

2.2 The Galilean Limit 5

2.3 Introduction of the Rapidity $\boldsymbol{\chi}$ 6

3 The Velocity of a Lorentz Transformation **8**

4 The General Lorentz Boost **10**

4.1 Quaternions and Pauli Spin Matrices 10

4.2 Lorentz Spinors and Transformation of the World Vector 12

4.3 Lorentz Boosts Represented by SL(2,$\mathbb{C}$) 17

4.4 Addition of Parallel Velocities 19

4.5 Lorentz Boosts Represented by SO(1,3) 20

4.6 Lorentz Boosts and the Lie Algebra $\mathfrak{so}(1,3)$ 23

5 The Thomas–Wigner Rotation **31**

5.1 Unified Representation of Boosts and Rotations 31

5.2 Polar Decomposition of Two Consecutive Boosts 32
5.3 Plausibility of the Equation System 34
5.4 Solving the Equation System 37
5.5 Addition of Non Parallel Velocities 42
5.6 Discussion of the Rotation Angle 44

Bibliography 47

1 The Foundation of Lorentz Transformations

In this chapter, we are going to gather some well–known information about Lorentz transformations, which we need to fall back on later. Apart from standard references like [5] and [7] we will also work with lecture notes on quantum field theory [8] from the RWTH Aachen. In its first chapter, this script presents a compact summary of the Lorentz group's essentials; it was offered for download in the year 2000:

Introduction Into Quantum Field Theory

1.1 Isometric Requirement

In Special Relativity, the space–time continuum gets its structure from the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.1}$$

This variant $\{+ - - -\}$ of the signature runs under the name "West Coast Metric" because it follows the conventions of the institutes of physics on the West Coast of the US. Lorentz transformations are homogeneous linear transformations ${\Lambda_\mu}^\nu$ which leave the Minkowski metric unaltered:

$$g_{\mu'\nu'} \;=\; {\Lambda_{\mu'}}^{\alpha}{\Lambda_{\nu'}}^{\beta} g_{\alpha\beta} \;:=\; g_{\mu\nu} \qquad \Rightarrow \qquad \Lambda\, g\, \Lambda^T \;=\; g. \tag{1.2}$$

In (1.2) we wrote down the "Isometric requirement" for Lorentz transformations in coordinates as well as in matrix form. From the latter we can derive limits for the determinant of Λ:

$$\det[\Lambda]\cdot\det[g]\cdot\det[\Lambda] \;=\; \det[g] \qquad \Rightarrow \qquad \det[\Lambda] = \pm 1. \tag{1.3}$$

Depending on the signature $\{p, q\}$ of the metric, linear transformations of the type (1.2) are either joined into the "Orthogonal group" O(p,q) or the "Special orthogonal group" SO(p,q), which includes all isometric transformations or only those with the determinant 1, respectively [1]. The term "Orthogonal" might be slightly misleading, because in general, these transformations are not orthogonal matrices. Only if we are dealing with the standard metric, which means $q = 0$ or $p = 0$, the transformation matrices actually are orthogonal:

$$\Lambda\Lambda^T = 1. \tag{1.4}$$

In this special case, we use the simplified terminology O(n) or SO(n) with $n = p + q$ being the dimension of the underlying space.

The orthogonal group for the Minkowski metric (1.1) is called the "Lorentz group", but we will limit ourselves to transformations with $\det[\Lambda] = 1$. Members of this subgroup don't change the direction of the time flow or how space is oriented; they form the "proper" Lorentz group SO(1,3) which also includes the "orthocronous" property (no time reversal) [8].

1.2 Polar Decomposition

Complex numbers $z = x + iy$ are members of the complex plane where x and y can be viewed as Cartesian coordinates. Alternatively, z can also be written in polar coordinates like

$$z \;=\; \mathrm{e}^{i\varphi} \cdot r\,. \tag{1.5}$$

The positive number r stands for the distance of the point z from the origin and φ is the angle relative to the real axis. The complex number $\mathrm{e}^{i\varphi}$ is an unimodular phase factor, and we can summarise the properties of r and $\mathrm{e}^{i\varphi}$ like so:

$$\mathrm{e}^{i\varphi}\,(\mathrm{e}^{i\varphi})^* = 1 \qquad \text{(unimodular)} \tag{1.6}$$

$$r \geq 0 \qquad \text{(positive).} \tag{1.7}$$

According to a theorem of matrix theory, the concept of polar coordinates can be transferred to invertible complex matrices. This is called the "Polar decomposition of a matrix", which implies that any such matrix A can be uniquely written as a product of a unitary matrix U and a positive Hermitian matrix H or H', respectively [8]:

$$A \;=\; UH \quad \text{or} \quad A \;=\; H'U. \tag{1.8}$$

Unitarity and Hermiticity are defined as

$$UU^\dagger = 1 \qquad \text{(unitary)} \tag{1.9}$$

$$H^\dagger = H \quad \text{or} \quad H'^{\dagger} = H' \qquad \text{(Hermitian).} \tag{1.10}$$

The symbol † merges complex conjugation ($*$) and transposition (T) into a single operation. The term "Positive Hermitian matrix" means that the eigenvalues of H or H' are positive or 0. The unitary part U of the polar decomposition may be the left or the right factor of the matrix product; the corresponding Hermitian matrices convert into each other like

$$H' = UHU^\dagger. \tag{1.11}$$

If we restrict the polar decomposition to real matrices, the unitary factor U will be replaced by an orthogonal matrix R, and the positive Hermitian factors H and H' will turn into positive symmetric matrices P and P':

$$A = RP \quad \text{or} \quad A = P'R. \tag{1.12}$$

Orthogonal and symmetric matrices fulfil conditions similar to (1.9) and (1.10):

$$RR^T = 1 \qquad\qquad \text{(orthogonal)} \tag{1.13}$$

$$P^T = P \quad \text{or} \quad P'^T = P' \qquad \text{(symmetric).} \tag{1.14}$$

Again, the term "Positive" implies that the real eigenvalues of the symmetric matrices P and P' are non–negative. Because of their position within the matrix product, P and P' are different and convert into each other like

$$P' = RPR^T. \tag{1.15}$$

Let us finally apply the polar decomposition to Lorentz transformations, which according to (1.12) yields

$$\Lambda = R\Lambda_B. \tag{1.16}$$

The matrix R rotates the coordinate system while Λ_B stands for a pure "Boost" transformation which transfers one inertial system into another without an additional rotation. The velocity vector of Λ_B will point in an arbitrary direction not necessarily parallel to a spacial coordinate axis. Both parts R and Λ_B are Lorentz transformations in their own right with the additional properties

$$RR^T = 1 \qquad \text{(rotation, orthogonal)} \tag{1.17}$$

$$\Lambda_B^T = \Lambda_B \qquad \text{(boost, symmetric).} \tag{1.18}$$

As a member of the proper Lorentz group SO(1,3), the rotation matrix R has to fulfil the isometric premise (1.2) and can't just be an arbitrary element of the group SO(4). It rather has to be a member of the subgroup SO(3), because equation (1.2) restricts its action via the metric's signature to the spacial coordinates. This causes R to break up into the following block structure:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_3 & \\ 0 & & & \end{pmatrix}, \qquad R_3 \in \mathrm{SO}(3). \tag{1.19}$$

The boost transformation Λ_B is also restricted by the isometric property because as a member of the proper Lorentz group, its determinant has to be $\det[\Lambda_B] = 1$. Hence, its eigenvalues are strictly positive and cannot vanish.

2 Derivation of the Lorentz Boost

A Lorentz transformation is called “Special” if the vector of the relative velocity between the two inertial frames coincides with a spacial axis. In this chapter, we will show that a special Lorentz transformation is already determined by the isometric requirement up to one parameter. This remaining parameter relates to the velocity v, which follows from an investigation of the Galilean limit $v/c \ll 1$.

Since a special Lorentz transformation just affects a single spacial coordinate, we can reduce the Minkowski space to two dimensions: x^0 (time) and x^1 (space). Every Lorentz transformation is a pure boost in this two–dimensional space–time because there are no rotations in just one spacial dimension. We label the matrix elements of the special Lorentz transformation Λ_S as a, b, c und d:

$$\Lambda_S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{2.1}$$

As we already know, this matrix has to be symmetric, which means that b or c could be eliminated. However, we would like to demonstrate that the symmetry already follows from the isometric requirement. The metric of this two–dimensional space–time continuum reads

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.2}$$

2.1 Solving the Equation System

Now let’s use our assumption (2.1) in the isometric premise $\Lambda_S \, g \, \Lambda_S^T = g$ which gives us

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & ac - bd \\ ac - bd & c^2 - d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.3}$$

This can be viewed as a system of three independent equations:

$$a^2 - b^2 = 1 \tag{2.4}$$

$$c^2 - d^2 = -1 \tag{2.5}$$

$$ac - bd = 0. \tag{2.6}$$

We will solve this system as follows:

$$b = (c/d)\,a \qquad \text{from (2.6)} \tag{2.7}$$

$$a^2\left(1-(c/d)^2\right) = 1 \qquad \text{from (2.4) and (2.7)} \tag{2.8}$$

$$a^2 = d^2 \qquad \text{from (2.8) and (2.5)} \tag{2.9}$$

$$b^2 = c^2 \qquad \text{from (2.9) and (2.6).} \tag{2.10}$$

Giving $c/d =: \beta$ an own designation we get

$$a^2(1-\beta^2) = 1 \qquad \text{from (2.8)} \tag{2.11}$$

$$b = \beta a \qquad \text{from (2.7).} \tag{2.12}$$

Equations (2.7) – (2.12) let us rewrite all coefficients by using β:

$$a = d = \frac{1}{\sqrt{1-\beta^2}} =: \gamma \qquad \text{from (2.11) and (2.9)} \tag{2.13}$$

$$b = c = \frac{\beta}{\sqrt{1-\beta^2}} =: \gamma\beta \qquad \text{from (2.12), (2.13) and (2.10).} \tag{2.14}$$

The introduction of γ in the above equations might appear superfluous, but it helps to keep the notation concise and the "Gamma factor" is well established in Special Relativity. Choosing the positive root when solving the squares ensures that we stay within the orthochronous Lorentz group. Elements of this matrix group have a positive component ${\Lambda_0}^0$ which keeps the time flow from getting reversed. Therefore, this is the special Lorentz transformation in matrix form:

$$\Lambda_S = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}. \tag{2.15}$$

2.2 The Galilean Limit

To determine how β relates to the relative velocity v, we will investigate how the Lorentz transformation affects the components of the world vector

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{2.16}$$

At this point we want to state that the special Lorentz transformation (2.15) does not transform components, but base vectors. For the transformation of components we have to use the inverse transposed matrix instead:

$$\tilde{\Lambda}_S = (\Lambda_S^{-1})^T = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}. \tag{2.17}$$

Because of this peculiar behaviour the components (2.16) are called "Contravariant". With $\tilde{\Lambda}_S$ we can carry out the component transformation:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \gamma \begin{pmatrix} ct - \beta\, x \\ x - \beta\, ct \end{pmatrix} \tag{2.18}$$

or

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} t - \beta/c\, x \\ x - \beta\, ct \end{pmatrix}. \tag{2.19}$$

Now we need to perform the limit $c \to \infty$ and compare the result to the corresponding Galilei transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} t \\ x - vt \end{pmatrix}. \tag{2.20}$$

In a first step, carrying out this limit in (2.19) yields

$$\lim_{c\to\infty} \begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} t \\ x - \beta\, ct \end{pmatrix}. \tag{2.21}$$

To ensure that the term βc does not diverge, we have to assume that β goes to zero like $\beta \sim 1/c$. As a result, we can skip the square root in (2.21) and directly compare the outcome with (2.20):

$$\lim_{c\to\infty} \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} t \\ x - \beta\, ct \end{pmatrix} := \begin{pmatrix} t \\ x - vt \end{pmatrix}. \tag{2.22}$$

The x–component of (2.22) determines β as

$$\beta = \frac{v}{c}. \tag{2.23}$$

2.3 Introduction of the Rapidity χ

Interestingly, a special Lorentz transformation can be understood as some kind of rotation about a quasi–angle χ which is called "Rapidity". However, the transformation matrix (2.15) of Λ_S is an element of the Lorentz group SO(1,1) and not of the rotation group SO(2), which also means it is not orthogonal. Instead, Λ_S follows some sort of anti–orthogonality relation

$$\Lambda_S \Lambda_S^{\overline{T}} = 1. \tag{2.24}$$

$\Lambda_S^{\overline{T}}$ stands for the transposed matrix in which additionally the sign of all non–diagonal elements is switched. The transformation matrix $\tilde{\Lambda}_S$ for contravariant tensor components already has this structure because it was derived from the inverse of Λ_S, see (2.15) and (2.17). This anti–orthogonality relation is only valid for special Lorentz transformations, it can't be extended to general boosts. The anti–orthogonality implies that the components of Λ_S are hyperbolic functions of the rapidity and not trigonometric ones:

$$\cosh(\chi) := \gamma \tag{2.25}$$

$$\sinh(\chi) := \gamma\beta. \tag{2.26}$$

With the above definitions, the functional relation for hyperbolic functions holds as it should:

$$\cosh^2(\chi) - \sinh^2(\chi) = \gamma^2(1 - \beta^2) = 1. \tag{2.27}$$

The rapidity can be calculated from equation (2.25) and (2.26):

$$\tanh(\chi) = \beta \quad \Rightarrow \quad \chi = \operatorname{artanh}\left(\frac{v}{c}\right). \tag{2.28}$$

The special Lorentz transformation 2.15 can now be expressed by χ:

$$\Lambda_S = \begin{pmatrix} \cosh(\chi) & \sinh(\chi) \\ \sinh(\chi) & \cosh(\chi) \end{pmatrix}, \tag{2.29}$$

which makes the analogy with a rotation matrix especially apparent.

3 The Velocity of a Lorentz Transformation

This chapter is about a simple method to extract the velocity or the parameter β in case a Lorentz transformation Λ is only known by its numerical values. The idea was adopted from the lecture notes [8].

In section 1.2 we learned about the polar decomposition which allows us to decompose a Lorentz transformation into two operations, a rotation R and a boost Λ_B:

$$\Lambda = R\,\Lambda_B. \tag{3.1}$$

The rotation R vanishes in the procedure shown below:

$$\Lambda^T\Lambda = \Lambda_B^T\,R^T R\,\Lambda_B = \Lambda_B^T\Lambda_B, \tag{3.2}$$

because R is orthogonal and thus $R^T R = 1$ applies. The next step would be to find a different coordinate system in which the direction of the velocity vector coincides with one of the coordinate axes. We arbitrarily choose axis x^1, which not only means we are allowed to limit our further investigations to a boost Λ_B, but we even can restrict ourselves to a special Lorentz transformation Λ_S. The base change is done by a rotation R_S, which itself is an orthogonal Matrix

$$R_S R_S^T = 1. \tag{3.3}$$

Λ_B and Λ_S transform into each other via

$$\Lambda_B = R_S^T \Lambda_S R_S \tag{3.4}$$

(redirect coordinates, apply the special Lorentz transformation, rotate coordinates back). Using the expression (3.4) for Λ_B in (3.2) yields

$$\Lambda^T\Lambda = \Lambda_B^T\Lambda_B = R_S^T\Lambda_S^T R_S R_S^T \Lambda_S R_S = R_S^T\Lambda_S^T\Lambda_S R_S. \tag{3.5}$$

The matrix R_S vanishes completely in the trace of (3.5):

$$\mathrm{Tr}[\Lambda^T\Lambda] = \mathrm{Tr}[R_S^T\Lambda_S^T\Lambda_S R_S] = \mathrm{Tr}[R_S R_S^T\Lambda_S^T\Lambda_S] = \mathrm{Tr}[\Lambda_S^T\Lambda_S]\,. \tag{3.6}$$

We can evaluate the trace using the explicit form of Λ_S

$$\Lambda_S = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.7}$$

which results in

$$\mathrm{Tr}[\Lambda^T\Lambda] = \mathrm{Tr}\begin{pmatrix} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.8}$$

$$= 2\left[\gamma^2(1+\beta^2)+1\right] = 2\gamma^2\left[1+\beta^2+1-\beta^2\right] = 4\gamma^2. \tag{3.9}$$

This relation together with the gamma factor's property $\gamma^2 = 1/(1-\beta^2)$ allows us to calculate the velocity from the trace:

$$\left(\frac{v}{c}\right)^2 = \beta^2 = 1 - \frac{1}{\gamma^2} = 1 - \frac{4}{\mathrm{Tr}[\Lambda^T\Lambda]}. \tag{3.10}$$

Finally, we would like to summarize how we can extract the parameters γ and β from an element Λ of the orthochronous proper Lorentz group if we just have its numerical components:

$$\gamma^2 = \frac{1}{4}\,\mathrm{Tr}[\Lambda^T\Lambda] \tag{3.11}$$

$$\beta^2 = 1 - \frac{4}{\mathrm{Tr}[\Lambda^T\Lambda]}. \tag{3.12}$$

4 The General Lorentz Boost

One particular characteristic of a special Lorentz transformation Λ_S is how it formally resembles a rotation in two spacial dimensions. In this analogy, the rotation angle corresponds to the rapidity χ

$$\tanh(\chi) = \beta = \frac{v}{c} \tag{4.1}$$

and all trigonometric functions in the rotation matrix are exchanged by their hyperbolic counterparts. In case the relative velocity corresponds to the x^1–axis, the special Lorentz transformation for contravariant components (see section 2.2) reads

$$\Lambda_S = \begin{pmatrix} \cosh(\chi) & -\sinh(\chi) & 0 & 0 \\ -\sinh(\chi) & \cosh(\chi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

In this chapter, we will investigate how we could generalise special Lorentz transformations Λ_S to arbitrary boosts Λ_B. With the latter, the velocity vector points in a random direction, whereas it coincides with a coordinate axis with special Lorentz transformations. This situation resembles the transition from rotations around a coordinate axis to arbitrary rotations [3]. Elegant though a little exotic is the representation of general rotations as quaternions. In physics, the interpretation of the rotation group SO(3) as a Lie group with an associated Lie algebra $\mathfrak{so}(3)$ is much more common. Among other things, the latter enables us to derive the rotation formula [2] of Olinde Rodrigues in a simple way.

Instead of starting off with quaternions, physicists tend to approach Lorentz transformations differently by mapping the world vector onto a complex hermitian 2×2–matrix. This class of matrices represents a four–dimensional real vector space which makes it isomorphic to Minkowski space. Using the Pauli spin matrices as a base, the world vector gets transformed into a complex hermitian 2×2– matrix, thus becoming a member of the matrix group GL(2,$\mathbb{C}$), as mentioned above. In the next sections, we will first explore some properties of the Pauli spin matrices, introduce Lorentz spinors and finally derive an explicit expression for a general Lorentz boost.

4.1 Quaternions and Pauli Spin Matrices

In our study [3] on rotations we already have talked about the Pauli spin matrices as matrix representations of the imaginary quaternions $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Quaternions can be seen as generalized complex numbers with three imaginary dimensions ($\mathbf{i}\,\mathbf{j}\,\mathbf{k}$) instead of only one ($i$). The three imaginary units multiply according to Hamilton's well–known multiplication rule

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \tag{4.3}$$

from which the following multiplication table can be derived:

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i} \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}\,. \end{aligned} \tag{4.4}$$

This table clearly shows that the multiplication of quaternions isn't commutative. The quaternion base is incomplete until we add the real unit 1 to the imaginary units **i**, **j** and **k**. A multiplication with 1 leaves any quaternion unaltered regardless if it is done from the right or the left side. Now, every quaternion **Q** can be written as a linear combination of the four base quaternions with real coefficients a, b, c, d:

$$\mathbf{Q} = a\,1 + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}. \tag{4.5}$$

The multiplication rules (4.4) of the imaginary base quaternions are so similar to those of the Pauli matrices $\boldsymbol{\sigma}_i$ that the latter may represent the former:

$$\mathbf{i} = -i\boldsymbol{\sigma}_1 \qquad \mathbf{j} = -i\boldsymbol{\sigma}_2 \qquad \mathbf{k} = -i\boldsymbol{\sigma}_3 \tag{4.6}$$

The Pauli matrices are three complex 2×2– matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.7}$$

which are hermitian, unitary and traceless, as one can easily check. They satisfy the relation

$$\boldsymbol{\sigma}_1{}^2 = \boldsymbol{\sigma}_2{}^2 = \boldsymbol{\sigma}_3{}^2 = \; -i\,\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_3 = \mathbb{1} \tag{4.8}$$

as well as the multiplication table

$$\begin{aligned} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 &= -\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_3 &= -\boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_1 &= -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_2. \end{aligned} \tag{4.9}$$

If we expand the set of Pauli matrices with unit matrix $\mathbb{1}$ which is hermitian and unitary but not traceless, we obtain a base of GL(2,$\mathbb{C}$), the space of complex invertible 2×2– matrices. Any matrix $\mathbf{S} \in$ GL(2,$\mathbb{C}$) can thus be written as

$$\mathbf{S} \;=\; a\mathbb{1} + b\,\boldsymbol{\sigma}_1 + c\,\boldsymbol{\sigma}_2 + d\,\boldsymbol{\sigma}_3 \tag{4.10}$$

with complex coefficients a, b, c and d. With real coefficients, the matrix **S** is hermitian.

4.2 Lorentz Spinors and Transformation of the World Vector

Before talking about Lorentz transformations and Lorentz spinors, we want to introduce the mathematical terminology and the necessary tools in the context of rotations. In physics, the term "Spinor" is used to describe how objects behave when they spin. As an example, in our text [3] on rotations we talked about things which – like the quantum states of a spin–1/2 particle – reach their initial state only after a rotation through 4π or 720°. We generally call anything a "Spinor" if we are interested in its rotational characteristics. From a mathematical point of view, spinors are elements of complex vector spaces with different dimensions. They are marked with an index which runs through the half–integers $j = 0, \frac{1}{2}, 1, \cdots$:

$$S_{(j)} : \; j\text{–spinor of dimension } 2j+1\,.$$

The index j just assigns a name to a spinor, it shouldn't be mistaken as an index of a tensor component or a summation index. To enable a rotation to act on a spinor $S_{(j)}$, it needs a matrix representative that operates in the spinor's $2j+1$–dimensional vector space. The entirety of those $(2j+1)\times(2j+1)$ – matrices stands for rotations in the vector space of the spinor $S_{(j)}$. It is called the representation $D^{(j)}$ of the rotation group:

$$D^{(j)} : \; \text{Representation of the rotation group in the space } \mathbb{C}^{2j+1}\,.$$

In [3] we used the matrix groups SO(3) and SU(2) as representations of the abstract rotation group. We will now apply spinor terminology to both of them:

– The elements of the group SO(3) act as rotation matrices on objects within the three–dimensional space $\mathbb{R}^3$ of position vectors. These objects are named "Real 1–spinors $S_{(1)}$", and the group SO(3) is a real version of the representation $D^{(1)}$.

– In quantum mechanics, the complex quantum states must be normalized, which means that rotations should be unitary operators. For example, the quantum states in the two–dimensional Hilbert space of a spin–$\frac{1}{2}$ particle are called "$\frac{1}{2}$–spinors $S_{(1/2)}$". In this Hilbert space, rotations are represented by elements of the group SU(2) which is named $D^{(1/2)}$.

The matrix groups SO(3) and SU(2) represent the abstract rotation group in the vector spaces $\mathbb{R}^3$ and $\mathbb{C}^2$. As representations, they are homomorphic regarding the group operation, which means that the multiplication of elements of a matrix representation will preserve the group structure. Regardless that both groups are representations of the rotation group and can be homomorphically mapped into each other, they are not isomorphic. They rather live in a two–to–one relationship inasmuch the positive and negative version of any element of SU(2) is mapped onto just one element of SO(3). This is why the group SU(2) is called a "Double Cover" of SO(3) [1], [3]. Obviously, the representation SU(2) captures some subtle aspects of rotations which slip through unnoticed in $\mathbb{R}^3$, see e.g. [4].

Picking up the actual topic of this chapter again, we are going to transfer the above–mentioned elements of representation theory from the realm of rotations to Lorentz transformations. The latter are acting upon things called "Lorentz Spinors", which extends the spinor concept of rotations in a way that it also applies to Lorentz transformations. For this purpose, we have to introduce matrix groups that are homomorphic to Lorentz transformations and can represent them like the groups SO(3) and SU(2) represent rotations:

- The special orthogonal group SO(3) gets replaced by the proper orthochronous Lorentz Group SO(1,3). Possibly included rotations can be separated by polar decomposition leaving the remainder symmetric. Examples are the special Lorentz transformations of Relativity, which are rotationless. On the other hand, pure rotations form a 3×3 – block within the 4×4 – matrix of a Lorentz transformation, see (1.19).
- The group SU(2) of unitary matrices gets extended to the group SL(2,$\mathbb{C}$), which is a surprisingly general and unrestricted set of complex 2×2 – matrices. Like with the Lorentz group SO(1,3), polar decomposition will separate the unitary parts (1.8), leaving a Hermitian remainder.

Unlike unitary groups like SU(2), working with the representation SL(2,$\mathbb{C}$) makes us take a closer look at the concepts of duality and adjunction with respect to the inner product of quantum states. Both affect the way how a transformation U acts upon one factor of the inner product of two quantum states $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$. Let's first write down the definitions:

$$\langle U^\dagger\beta \mid \alpha\rangle := \langle\beta \mid U\alpha\rangle \qquad \text{(adjoint transformation)} \tag{4.11}$$

$$\langle U^{(*)}\beta \mid \alpha\rangle := \langle\beta \mid U^{-1}\alpha\rangle \qquad \text{(dual transformation)} \tag{4.12}$$

For the transformations U and U^{-1}, which act upon $|\alpha\rangle$, we are looking for corresponding transformations $U^\dagger$ and $U^{(*)}$, which act upon $|\beta\rangle$ and leave the inner product unaltered. In (4.11), the adjoint operator $U^\dagger$ is defined by the structure of the inner product of quantum states. In (4.12), however, this inner product is understood as a linear functional which is parameterized by $|\beta\rangle$ and acts upon $|\alpha\rangle$. In this context, U is treated as an active transformation of the system. By expanding the states $|\alpha\rangle$ and $|\beta\rangle$ in an Eigenbase of any quantum operator it is easy to evaluate the adjoint and dual transformations of U explicitly [3]:

$$U^\dagger = U^{*T} \tag{4.13}$$

$$U^{(*)} = (U^{-1})^{*T} \tag{4.14}$$

For unitary transformations $UU^\dagger = \mathbb{1}$ we don't have to pay attention to dual transformations because

$$\langle U^{(*)}\beta \mid \alpha\rangle \;=\; \langle \beta \mid U^{-1}\alpha\rangle \;=\; \langle (U^{-1})^\dagger \beta \mid \alpha\rangle \;=\; \langle U\beta \mid \alpha\rangle\,. \tag{4.15}$$

According to the first and the last term in (4.15), inversion and adjunction cancel each other making $U^{(*)} = U$ self–dual. The difference between regular and dual transformations is a complex variant of the distinction between contravariant and covariant coordinates. The latter also vanishes if the transformation is orthogonal, which is unitary but restricted to real numbers. The meaning of equation (4.15) (the second and the last term) becomes apparent if we visualize U as a rotation: The value of the inner product depends on the relative position of the vectors $|\alpha\rangle$ and $|\beta\rangle$. It doesn't matter if $|\beta\rangle$ is turned "ahead" or $|\alpha\rangle$ is turned "back", the inner product just depends on the relative position of the vectors and the outcome is the same in both cases.

In $\mathbb{C}^2$, Lorentz transformations are not represented by the matrix group SU(2) but by the group SL(2,$\mathbb{C}$), whose elements L are not self–dual:

$$\langle L^{(*)}\beta \mid \alpha\rangle \;=\; \langle \beta \mid L^{-1}\alpha\rangle \;=\; \langle (L^{-1})^\dagger \beta \mid \alpha\rangle \;\neq\; \langle L\,\beta \mid \alpha\rangle\,. \tag{4.16}$$

In what follows, we must not only be able to properly transform spinor–kets, but also their bra variants. To accomplish this, we need in addition to ordinary Lorentz transformations their adjoint counterparts, i.e. their transposed and conjugate complex versions. We don't have to care about transposition which automatically comes with matrix multiplications and the position of the factors. The complex conjugation, however, must be carried out explicitly. We need to keep track in the Ricci calculus which spinor index transforms how, and therefore, every bra–related index is marked with an additional dot. This provides us with two variants of Lorentz transformations:

$L_\alpha{}^\beta$: Ordinary Lorentz transformation for kets

$L^*_{\dot\alpha}{}^{\dot\beta}$: Conjugate Lorentz transformation for bras

The introduction of conjugate Lorentz transformations and dotted indices requires the spinor– and representation terminology to be slightly altered:

– Lorentz spinors are now viewed as tensor products of an ordinary j–spinor and a conjugate k–spinor. The indices run separately through the half–integer values $0, \frac{1}{2}, 1, \cdots$:

  $S_{(jk)}$: j, k–Lorentz spinor of dimension $(2j+1)(2k+1)$.

  Pure ordinary or conjugate spinors get a 0 as a second index, respectively:

  $S_{(j0)}$: Ordinary $j, 0$–Lorentz spinor of dimension $(2j+1)$

  $S_{(0k)}$: Conjugate $0, k$–Lorentz spinor of dimension $(2k+1)$.

– A Lorentz transformation $L \in \mathrm{SL}(2,\mathbb{C})$ acting upon a spinor $S_{(jk)}$ needs a representation $D^{(jk)}$ in the spinor's $(2j+1)(2k+1)$–dimensional Hilbert space. Moreover, $D^{(jk)}$ is regarded as a representation of the group SL(2,$\mathbb{C}$) itself, which then takes the role of an abstract group that has representations in spinor spaces of various dimensions. There are two fundamental representations of SL(2,$\mathbb{C}$) in the Hilbert space $\mathbb{C}^2$:

$D^{(\frac{1}{2}0)}$: The members of the group L itself with components ${L_\alpha}^\beta$

$D^{(0\frac{1}{2})}$: The conjugate version of L with components ${L^*_{\dot\alpha}}^{\dot\beta}$.

In our paper [3] on rotations we found that the Pauli matrices $\boldsymbol{\sigma}_i$ let us map any position vector $\underline{x} \in \mathbb{R}^3$ onto a trace–free Hermitian matrix:

$$\underline{x} \;\to\; \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} = x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2 + x^3\boldsymbol{\sigma}_3\,. \tag{4.17}$$

When we apply a similarity transformation to this position vector using a unitary matrix

$$\mathbf{x}' = U\,\mathbf{x}\,U^{-1} \qquad U \in \mathrm{SU}(2)\,, \tag{4.18}$$

the following properties apply:

– The result $\mathbf{x}'$ is another trace–free Hermitian matrix and can be interpreted as a position vector as well.

– The determinant $\det[\mathbf{x}] = (x^1)^2 + (x^2)^2 + (x^3)^2$ doesn't change under a similarity transformation, which means that the Euclidean length of $\underline{x}$ is preserved.

With this in mind, we can conclude that the unitary matrix U acts upon x like a rotation. The position vector $\underline{x}$, from which the matrix x has been derived, is a member of $\mathbb{R}^3$ and transforms like a (1)–spinor. In [3] we used the transformation rule (4.18) as a blueprint for the spin $-\,1$ representation of the group SU(2) or the abstract rotation group in $\mathbb{R}^3$. The transformation (4.18) enabled us to construct a rotation matrix $R \in \mathrm{SO}(3)$ from the unitary matrix $U \in \mathrm{SU}(2)$.

With this knowledge, the next logical step would be to move on to Lorentz transformations and turn the position vector into a world vector by adding the matrix $\boldsymbol{\sigma}_0 = \mathbb{1}$ to the Pauli matrices $\boldsymbol{\sigma}_i$:

$$\mathbf{x} \;\to\; \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x^0\boldsymbol{\sigma}_0 + x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2 + x^3\boldsymbol{\sigma}_3 \;=\; x^\mu\boldsymbol{\sigma}_\mu\,. \tag{4.19}$$

Four types of Lorentz spinors are four–dimensional and could theoretically apply for a description of the world vector (4.19): $(0,\frac{3}{2})$, $(\frac{3}{2},0)$ und $(\frac{1}{2},\frac{1}{2})$. In (4.19) we interpreted the world vector as a Hermitian 2×2–matrix, which picks out the representation $(\frac{1}{2},\frac{1}{2})$. This also determines its transformation behaviour, and the Lorentz transformation $L \in \mathrm{SL}(2,\mathbb{C})$ acts upon $\mathbf{x}$ like

$$\mathbf{x}'_{\mu\dot{\nu}} \;=\; \mathbf{x}_{\alpha\dot{\beta}}\, L_{\mu}{}^{\alpha}\, L^{*}_{\dot{\nu}}{}^{\dot{\beta}} \qquad \Rightarrow \qquad \mathbf{x}' \;=\; L\,\mathbf{x}\,L^{\dagger}\,. \tag{4.20}$$

The above transformation is alternatively written in the Ricci calculus or with matrix operations. In the special case of a pure rotation the transformation matrix $L^{\dagger} = L^{-1}$ is unitary and (4.20) reduces to the similarity transformation (4.18). The characteristics of Lorentz transformations stay preserved under (4.20):

– The transformed matrix $\mathbf{x}'$ is again Hermitian and can still be interpreted as a world vector:

$$\mathbf{x}'^{\dagger} \;=\; L^{\dagger\dagger}\,\mathbf{x}^{\dagger}L^{\dagger} \;=\; L\,\mathbf{x}\,L^{\dagger} \;=\; \mathbf{x}'. \tag{4.21}$$

– The determinant of the transformed world vector matrix $\mathbf{x}'$ doesn't change because $L \in \mathrm{SL}(2,\mathbb{C})$, which means $\det[L] = 1$.

$$\det[\mathbf{x}'] \;=\; \det[L]\cdot\det[\mathbf{x}]\cdot\det[L^{\dagger}] \;=\; |\det[L]|^{2}\cdot\det[\mathbf{x}] \;=\; \det[\mathbf{x}]. \tag{4.22}$$

Explicit calculation shows that the length of the world vector stands for the invariant proper time:

$$\begin{aligned}\det[\mathbf{x}] &= (x^0 + x^3)(x^0 - x^3) \;-\; (x^1 + ix^2)(x^1 - ix^2)\\ &= \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 .\end{aligned} \tag{4.23}$$

This makes the representation $D^{(\frac{1}{2},\frac{1}{2})}$ isometric concerning the Minkowski metric, which allows us to consider it a Lorentz transformation in the first place.

– The trace of the world vector matrix $\mathrm{Tr}\,[\mathbf{x}] = 2x^0$ is not preserved under (4.20). It doesn't have to, because the time component of the world vector is no Lorentz scalar.

In this chapter, we extended the homomorphism between the groups SU(2) and SO(3) which represent rotations inasmuch as it now involves the groups SL(2,$\mathbb{C}$) and SO(1,3). As with the case of pure rotations, this homomorphism preserves the structure of the group operation, but it is not an isomorphism. It rather is a two–to–one relation which assigns every element of the group SO(1,3) the positive and negative version of an element of SL(2,$\mathbb{C}$). This kind of relation is called a "Double Cover".

4.3 Lorentz Boosts Represented by SL(2,$\mathbb{C}$)

As an element of the group SL(2,$\mathbb{C}$), the matrix L is invertible and can uniquely be factorized into a unitary part $U \in$ SU(2) and a Hermitian part L_B by polar decomposition. In (1.16) we already did this for $\Lambda \in$ SO(1,3), and now we will apply it on $L \in$ SL(2,$\mathbb{C}$):

$$
\begin{aligned}
&L = UL_B && \text{(corresponds to } \Lambda \in \mathrm{SO(1,3)})\\
&UU^\dagger = 1 && \text{(corresponds to } R \in \mathrm{SO(3)})\\
&L_B^\dagger = L_B && \text{(corresponds to } \Lambda_B \in \mathrm{SO(1,3)}).
\end{aligned} \tag{4.24}
$$

The unitary matrix $U \in$ SU(2) is an element of the representation $D^{(1/2)}$, which implements an abstract rotation in the space of 1/2–spinors. Actually, in our paper [3] on rotations, this representation was an early result that came from the introduction of angle–axis coordinates to parameterize rotations. A natural consequence of this parameterization is to express rotations as quaternions, which for their part can be rewritten as unitary matrices $U \in$ SU(2) with the help of the Pauli matrices (4.6). Based on this, the representation $D^{(1)}$ (a.k.a elements of the group SO(3)) was constructed in a following step.

Both Hermitian (4.10) and unitary matrices can be expanded in the basis of Pauli matrices. For the latter, a is real, b, c and d are purely imaginary, and the absolute squares of all four components sum up to 1. The boost part L_B of a Lorentz transformation however is Hermitian and its expansion coefficients are all real. Both representations of U and L_B are so similar that we will transfer the method discussed in [3] onto Lorentz boosts. In doing so we will replace the angle of rotation with the rapidity χ and present our ansatz for L_B in comparison to the formula for unitary matrices U which we know from [3]:

$$
\begin{aligned}
U &= \mathrm{e}^{-i(\theta/2)\mathbf{n}} = \cos(\theta/2)\,\boldsymbol{\sigma}_0 - i\sin(\theta/2)\,\mathbf{n}\\
L_B &= \mathrm{e}^{-(\chi/2)\mathbf{n}} = \cosh(\chi/2)\,\boldsymbol{\sigma}_0 - \sinh(\chi/2)\,\mathbf{n}.\\
\mathbf{n} &= n^1\boldsymbol{\sigma}_1 + n^2\boldsymbol{\sigma}_2 + n^3\boldsymbol{\sigma}_3
\end{aligned} \tag{4.25}
$$

The expression $\mathbf{n}$ stands for the unit vector which points in the direction of the rotation axis or the relative velocity, depending on whether it is a rotation or a Lorentz transformation. Similar to Olinde Rodrigues' use of the half–angle in his description of rotations [2], we introduced the "half–rapidity" $\chi/2$ in (4.25). Thereby a Lorentz boost is parameterized with the following quantities:

χ : Rapidity of the boost
n^1 : x–component of the normal vector $\underline{n}$ of the relative velocity
n^2 : y–component of the normal vector $\underline{n}$ of the relative velocity
n^3 : z–component of the normal vector $\underline{n}$ of the relative velocity.

The exponential function in (4.25) should be understood as an expansion of the operator **n** in a power series that combines the even powers into cosh and the odd powers into sinh. This is the counterpart to Euler's formula which should be used when two boosts are concatenated. The reason behind this is that generally the operators associated with the normal vectors of two boosts don't commute, which means the well–known exponential laws would have to be replaced by the Baker– Campbell– Hausdorff formula. Comparing (4.25) and (4.10), we can determine the coefficients for a Lorentz boost:

$$a = \cosh(\chi/2) \quad b = -\sinh(\chi/2)n^1 \quad c = -\sinh(\chi/2)n^2 \quad d = -\sinh(\chi/2)n^3. \tag{4.26}$$

The negative sign is related to the one in the coefficients (4.2) of the special Lorentz transformation. The components of $\underline{n}$ are normalized which leads to

$$\left(n^1\right)^2 + \left(n^2\right)^2 + \left(n^3\right)^2 = 1 \quad \Rightarrow \quad a^2 - b^2 - c^2 - d^2 = 1. \tag{4.27}$$

As a Hermitian element of the group SL(2,ℂ), L_B can be parameterized like

$$L_B = \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix}, \tag{4.28}$$

which makes (4.27) a condition for its determinant which, as expected, turns out to be

$$\det[L_B] \;=\; a^2 - b^2 - c^2 - d^2 \;=\; 1\,. \tag{4.29}$$

The properties of the operator **n** are similar to those of the normal vector $\underline{n}$ from which it was derived:

$$\begin{aligned} \mathbf{n}^2 &= \left(n^1\boldsymbol{\sigma}_1 + n^2\boldsymbol{\sigma}_2 + n^3\boldsymbol{\sigma}_3\right)^2 \\ &= \left[\left(n^1\right)^2 + \left(n^2\right)^2 + \left(n^3\right)^2\right]\boldsymbol{\sigma}_0 + \cdots \\ \cdots &+ i\left[n^1n^2 - n^2n^1\right]\boldsymbol{\sigma}_3 + i\left[n^3n^1 - n^1n^3\right]\boldsymbol{\sigma}_2 + i\left[n^2n^3 - n^3n^2\right]\boldsymbol{\sigma}_1 \\ &= \mathbb{1}. \end{aligned} \tag{4.30}$$

Finally, we will show that the boost transformation with rapidity $-\chi$ is the inverse of the boost with the rapidity χ:

$$L_B(\chi)L_B(-\chi) \;=\; \mathrm{e}^{-(\chi/2)\mathbf{n}}\,\mathrm{e}^{(\chi/2)\mathbf{n}} \;=\; \mathrm{e}^{-[(\chi/2)-(\chi/2)]\mathbf{n}} \;=\; \mathbb{1}. \tag{4.31}$$

As the operator **n** commutates with itself, the well–known exponential laws are applicable in this case.

4.4 Addition of Parallel Velocities

In the theory of relativity, the addition of two velocities implies the concatenation of two Lorentz boosts. We have to keep in mind that boost transformations don't form a subgroup within the proper Lorentz group because they are represented by symmetric matrices in SO(1,3) or by Hermitian matrices in SL(2,$\mathbb{C}$). The product of two symmetric or two Hermitian matrices does not necessarily lead to another symmetric or Hermitian matrix, which becomes obvious considering two Hermitian matrices A and B:

$$(AB)^\dagger \;=\; B^\dagger A^\dagger \;=\; BA \neq AB. \tag{4.32}$$

We see that the product AB is only Hermitian if A and B commute. As a consequence, the matrix multiplication of symmetric or Hermitian matrices isn't a closed group operation. In other words, the concatenation of two boosts generally doesn't result in another boost. However, it is always possible to apply a polar decomposition (1.8) to the result and interpret it as a product of a Lorentz boost and an additional rotation:

$$\begin{aligned} \Lambda_B(\underline{n}_2,\chi_2)\,\Lambda_B(\underline{n}_1,\chi_1) \;&=\; R(\underline{n}_{\mathrm{R}},\theta)\,\Lambda_B(\underline{n}_{\mathrm{B}},\chi) \\ L_B(\underline{n}_2,\chi_2)\,L_B(\underline{n}_1,\chi_1) \;&=\; U(\underline{n}_{\mathrm{R}},\theta)\,L_B(\underline{n}_{\mathrm{B}},\chi), \end{aligned} \tag{4.33}$$

depending on whether the Lorentz group is represented by the matrix group SO(1,3) or SL(2,$\mathbb{C}$). In (4.33) we made use of the following parameters:

$\underline{n}_1$: Normal vector of the velocity of the first boost
$\underline{n}_2$: Normal vector of the velocity of the second boost
$\underline{n}_{\mathrm{B}}$: Normal vector of the velocity of the resulting boost
$\underline{n}_{\mathrm{R}}$: Normal vector of the axis of the additional rotation
χ_1 : Rapidity of the first boost
χ_2 : Rapidity of the second boost
χ : Rapidity of the resulting boost
θ : Angle of the additional rotation.

The rotation $R(\underline{n}_{\mathrm{R}},\theta)$ or $U(\underline{n}_{\mathrm{R}},\theta)$ is named "Thomas–Wigner Rotation" after Llewellyn Thomas and Eugene Wigner [9], [10]. Derived from this is the more familiar term "Thomas Precession", which appears in spin dynamics and describes a rotation caused by the continuous concatenation of differential Lorentz boosts.

We will now take a look at the special case of two boosts along the same direction which are represented by commuting matrices. Therefore, the resulting Lorentz transformation is again a pure boost without an additional Thomas–Wigner rotation. We can use the exponential functions (4.25) directly because the operator $\mathbf{n}$, which represents the normal vector $\underline{n}$, certainly commutes with itself:

$$L_B(\underline{n},\chi_2)\,L_B(\underline{n},\chi_1) \;=\; \mathrm{e}^{-(\chi_2/2)\mathbf{n}}\,\mathrm{e}^{-(\chi_1/2)\mathbf{n}} \;=\; \mathrm{e}^{-[(\chi_1/2)+(\chi_2/2)]\mathbf{n}} \;=\; L_B(\underline{n},\chi_1{+}\chi_2). \tag{4.34}$$

We see that the concatenation of two boosts along the same direction in fact means adding their rapidities:

$$\chi \;=\; \chi_1 + \chi_2. \tag{4.35}$$

According to the addition theorem of the hyperbolic tangent we get

$$\tanh(\chi) \;=\; \tanh(\chi_1 + \chi_2) \;=\; \frac{\tanh(\chi_1) + \tanh(\chi_2)}{1 + \tanh(\chi_1)\tanh(\chi_2)}\,, \tag{4.36}$$

which leads us to the addition theorem of the β factor (4.1) or the velocity v known from Special Relativity:

$$\beta \;=\; \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \qquad \Rightarrow \qquad v \;=\; \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\,. \tag{4.37}$$

The relativistic addition of velocities shown above starts from general Lorentz transformations (4.25) and thus expands the more elementary textbook approach [5] which is based on special Lorentz transformations. We have introduced (4.25) as an ansatz which so far is only justified by its formal analogy to rotations and that it preserves Hermiticity and the determinant of the position vector matrix. With the addition theorem of velocities following from it, its plausibility is further enhanced.

4.5 Lorentz Boosts Represented by SO(1,3)

One interesting point in our discussion of rotations [3] was the build–up of a rotation matrix $R \in \mathrm{SO}(3)$ for an arbitrary rotation. In this section we will do the same for an arbitrary Lorentz boost $\Lambda_B \in \mathrm{SO}(1{,}3)$. For this purpose, we will transform the components of the world vector (4.19) with L_B, from which we can read off the components of the matrix Λ_B. For our calculations, we will make use of the general form (4.28) of L_B which will lead us to a matrix Λ_B which resembles the structure of the general rotation in [3].

In the first step, we will determine the product $\mathbf{x}L_B^\dagger$, where we have to bear in mind that a boost transformation is Hermitian which means $L_B^\dagger = L_B$. Indexed components will help us to express our computations more elegant and concisely:

$$\{a^\mu\} := \{a, b, c, d\}. \tag{4.38}$$

As usual, Greek indices run through the values $\{0, 1, 2, 3\}$ and Latin ones through $\{1, 2, 3\}$.

$$\begin{aligned} \mathbf{x}L_B^\dagger \;&=\; \big(x^\mu \boldsymbol{\sigma}_\mu\big)\big(a^\mu \boldsymbol{\sigma}_\mu\big) \;=\; \big(x^0\boldsymbol{\sigma}_0 + x^i\boldsymbol{\sigma}_i\big)\big(a^0\boldsymbol{\sigma}_0 + a^i\boldsymbol{\sigma}_i\big) \\ &=\; x^0 a^0\,\boldsymbol{\sigma}_0 + x^0\big(a^i\boldsymbol{\sigma}_i\big) + a^0\big(x^i\boldsymbol{\sigma}_i\big) + \big(x^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big). \end{aligned} \tag{4.39}$$

We will still keep time– and space indices separate when we determine $\mathbf{x}'$:

$$
\begin{aligned}
\mathbf{x}' \;&=\; L_{B}\,\mathbf{x}L_{B}^{\dagger} \\
&=\; x^0\big(a^0\big)^2\boldsymbol{\sigma}_0 + x^0a^0\big(a^i\boldsymbol{\sigma}_i\big) + \big(a^0\big)^2\big(x^i\boldsymbol{\sigma}_i\big) + a^0\big(x^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big) + \cdots \\
&\quad \cdots + x^0a^0\big(a^i\boldsymbol{\sigma}_i\big) + x^0\big(a^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big) + a^0\big(a^i\boldsymbol{\sigma}_i\big)\big(x^i\boldsymbol{\sigma}_i\big) + \big(a^i\boldsymbol{\sigma}_i\big)\big(x^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big).
\end{aligned}
\tag{4.40}
$$

Now we expand the brackets and write down every term explicitly thus falling back to the non–indexed notation. The results below are easily comprehensible:

$$
\big(a^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big) \;=\; \big[b^2+c^2+d^2\big]\boldsymbol{\sigma}_0 \tag{4.41}
$$

$$
\begin{aligned}
\big(x^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big) \;&=\; \big[bx^1+cx^2+dx^3\big]\boldsymbol{\sigma}_0 + \cdots \\
&\quad \cdots + i\big[x^1c - x^2\,b\big]\boldsymbol{\sigma}_3 + i\big[x^3\,b - x^1d\big]\boldsymbol{\sigma}_2 + i\big[x^2d - x^3c\big]\boldsymbol{\sigma}_1
\end{aligned}
\tag{4.42}
$$

$$
\begin{aligned}
\big(a^i\boldsymbol{\sigma}_i\big)\big(x^i\boldsymbol{\sigma}_i\big) \;&=\; \big[bx^1+cx^2+dx^3\big]\boldsymbol{\sigma}_0 + \cdots \\
&\quad \cdots + i\big[bx^2 - cx^1\big]\boldsymbol{\sigma}_3 + i\big[dx^1 - bx^3\big]\boldsymbol{\sigma}_2 + i\big[cx^3 - dx^2\big]\boldsymbol{\sigma}_1
\end{aligned}
\tag{4.43}
$$

$$
\begin{aligned}
\big(a^i\boldsymbol{\sigma}_i\big)\big(x^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big) \;&=\; \big[\big(b^2-c^2-d^2\big)x^1 + 2bc\,x^2 + 2bd\,x^3\big]\boldsymbol{\sigma}_1 + \cdots \\
\cdots +&\;\; \big[\big(c^2-b^2-d^2\big)x^2 + 2bc\,x^1 + 2cd\,x^3\big]\boldsymbol{\sigma}_2 + \cdots \\
\cdots +&\;\; \big[\big(d^2-b^2-c^2\big)x^3 + 2bd\,x^1 + 2cd\,x^2\big]\boldsymbol{\sigma}_3.
\end{aligned}
\tag{4.44}
$$

Only the sum of (4.42) and (4.43) is needed for (4.40), which means that all imaginary parts vanish:

$$
\big(x^i\boldsymbol{\sigma}_i\big)\big(a^i\boldsymbol{\sigma}_i\big) + \big(a^i\boldsymbol{\sigma}_i\big)\big(x^i\boldsymbol{\sigma}_i\big) \;=\; 2\big[bx^1+cx^2+dx^3\big]\boldsymbol{\sigma}_0. \tag{4.45}
$$

As a consequence, summing up all parts of (4.40) yields a real result:

$$
\begin{aligned}
\mathbf{x}' \;=\; L_{B}\,\mathbf{x}L_{B}^{\dagger} \;&=\; x'^0\boldsymbol{\sigma}_0 + x'^1\boldsymbol{\sigma}_1 + x'^2\boldsymbol{\sigma}_2 + x'^3\boldsymbol{\sigma}_3 \\
&=\; \big[\big(a^2+b^2+c^2+d^2\big)x^0 + 2ab\,x^1 + 2ac\,x^2 + 2ad\,x^3\big]\boldsymbol{\sigma}_0 + \cdots \\
\cdots +&\;\; \big[\big(a^2+b^2-c^2-d^2\big)x^1 + 2ba\,x^0 + 2bc\;x^2 + 2bd\,x^3\big]\boldsymbol{\sigma}_1 + \cdots \\
\cdots +&\;\; \big[\big(a^2-b^2+c^2-d^2\big)x^2 + 2ca\,x^0 + 2cb\;x^1 + 2cd\,x^3\big]\boldsymbol{\sigma}_2 + \cdots \\
\cdots +&\;\; \big[\big(a^2-b^2-c^2+d^2\big)x^3 + 2da\,x^0 + 2db\,x^1 + 2dc\,x^2\big]\boldsymbol{\sigma}_3.
\end{aligned}
\tag{4.46}
$$

Finally, we can read off the components of Λ_{B} from equation (4.46):

$$
\Lambda_{B} \;=\; \begin{pmatrix}
a^2+b^2+c^2+d^2 & 2ab & 2ac & 2ad \\
2ab & a^2+b^2-c^2-d^2 & 2bc & 2bd \\
2ac & 2bc & a^2-b^2+c^2-d^2 & 2cd \\
2ad & 2bd & 2cd & a^2-b^2-c^2+d^2
\end{pmatrix}
\tag{4.47}
$$

The parameters a, b, c and d depend on the rapidity χ and the components of the direction $\underline{n}$ of the velocity vector:

$$a = \cosh(\chi/2) \quad b = -\sinh(\chi/2)n^1 \quad c = -\sinh(\chi/2)n^2 \quad d = -\sinh(\chi/2)n^3. \tag{4.48}$$

A more commonly known form of Λ_B has the rapidity χ expressed by the factors γ and β:

$$\cosh(\chi) = \gamma \tag{4.49}$$

$$\sinh(\chi) = \gamma\beta. \tag{4.50}$$

For a derivation, we will use some identities of hyperbolic functions that help to switch between half angle and full angle:

$$\cosh^2(\chi/2) \;-\; \sinh^2(\chi/2) \;=\; 1 \tag{4.51}$$

$$\cosh^2(\chi/2) \;+\; \sinh^2(\chi/2) \;=\; \cosh(\chi) \tag{4.52}$$

$$2\cosh(\chi/2)\sinh(\chi/2) \;=\; \sinh(\chi) \tag{4.53}$$

$$2\sinh^2(\chi/2) \;=\; \cosh(\chi) - 1 \tag{4.54}$$

The first rule (4.51) already establishes a constraint for the components a, b, c and d:

$$a^2 - b^2 - c^2 - d^2 = 1\,. \tag{4.55}$$

This allows us to rewrite the diagonal elements of Λ_B in a more compact form:

$$\Lambda_B \;=\; \begin{pmatrix} a^2+b^2+c^2+d^2 & 2ab & 2ac & 2ad \\ 2ab & 1+2b^2 & 2bc & 2bd \\ 2ac & 2bc & 1+2c^2 & 2cd \\ 2ad & 2bd & 2cd & 1+2d^2 \end{pmatrix}. \tag{4.56}$$

Using expressions (4.48) for a, b, c and d and recalculating them as functions of the full angle we can write some components of the boost matrix $[\Lambda_B]^\mu{}_\nu$ like so:

$$[\Lambda_B]^0{}_0 = \; a^2+b^2+c^2+d^2 \;=\; \cosh(\chi) \;=\; \gamma \tag{4.57}$$

$$[\Lambda_B]^1{}_1 = \; 1+2b^2 \;=\; 1+(\cosh(\chi)-1)\,{n^1}^2 \;=\; 1+(\gamma-1)\,{n^1}^2 \tag{4.58}$$

$$[\Lambda_B]^0{}_1 = \; 2ab \;=\; -\sinh(\chi)\,n^1 \;=\; -\gamma\beta\,n^1 \tag{4.59}$$

$$[\Lambda_B]^1{}_2 = \; 2bc \;=\; (\cosh(\chi)-1)\,n^1n^2 \;=\; (\gamma-1)\,n^1n^2\,. \tag{4.60}$$

The remaining components are easily computed in the same way, and the matrix of the Lorentz boost written in γ and β reads

$$\Lambda_B \;=\; \begin{pmatrix} \gamma & -\gamma\beta\,n^1 & -\gamma\beta\,n^2 & -\gamma\beta\,n^3 \\ -\gamma\beta\,n^1 & (\gamma-1)\,{n^1}^2+1 & (\gamma-1)\,n^1n^2 & (\gamma-1)\,n^1n^3 \\ -\gamma\beta\,n^2 & (\gamma-1)\,n^1n^2 & (\gamma-1)\,{n^2}^2+1 & (\gamma-1)\,n^2n^3 \\ -\gamma\beta\,n^3 & (\gamma-1)\,n^1n^3 & (\gamma-1)\,n^2n^3 & (\gamma-1)\,{n^3}^2+1 \end{pmatrix}. \tag{4.61}$$

Finally, we want to check that velocities pointing along the coordinate axes actually reproduce the correct special Lorentz transformations. For the Lorentz transformation along the spacial axis i we have

$$n^i = 1, \qquad n^k = 0 \quad (\text{für } k \neq i). \tag{4.62}$$

We label these special Lorentz transformations as $\Lambda_S^{(i)}$ and get

$$\Lambda_S^{(1)} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\chi) & -\sinh(\chi) & 0 & 0 \\ -\sinh(\chi) & \cosh(\chi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.63}$$

$$\Lambda_S^{(2)} = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\chi) & 0 & -\sinh(\chi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh(\chi) & 0 & \cosh(\chi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.64}$$

$$\Lambda_S^{(3)} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh(\chi) & 0 & 0 & -\sinh(\chi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\chi) & 0 & 0 & \cosh(\chi) \end{pmatrix}. \tag{4.65}$$

Apparently, the general Lorentz transformation (4.61) contains the special Lorentz transformations $\Lambda_S^{(i)}$ as a special case, which we understand as another confirmation of our ansatz (4.25).

4.6 Lorentz Boosts and the Lie Algebra $\mathfrak{so}(1,3)$

As soon as we delve more deeply into symmetry operations like rotations or Lorentz transformations, we can't avoid dealing with the theory of Lie groups. For example, in our paper [3] on rotations we learned that the Lie algebra $\mathfrak{so}(3)$ comes in handy for the derivation of the rotation formula of Olinde Rodrigues. In this section, we want to apply this method to the Lorentz group SO(1,3) and find out what the counterpart of Olinde Rodrigues' formula looks like for Lorentz boosts.

We will start compiling a list of criteria that the matrix members of SO(1,3) and $\mathfrak{so}(1,3)$ will have to meet. We have already mentioned the fundamental isometric requirement (1.2) which the Minkowski metric imposes on the elements Λ of the group SO(1,3):

$$\Lambda\, g\, \Lambda^T = g. \tag{4.66}$$

A Lie group relates to its Lie algebra like a differential manifold does to one of its tangent vector spaces. Every point in a manifold or Lie group is determined by its local properties, which reversely means that the global properties of the group can be found in every point, too. Particularly and without limiting generality, it is possible to restrict our investigations to the environment of the Lie group's identity element $\Lambda = \mathbb{1}$, which then gets the zero element of the Lie algebra attached to it. It might be helpful to visualize this process by attaching a tangent plane to one particular surface point of a sphere. Having this in mind, we are going to look at a continuously differentiable path $\Lambda(t)$ in the group of Lorentz transformations of the form

$$\Lambda(t) \,=\, \mathrm{e}^{tM} \quad \text{with} \quad t \in \mathbb{R}\,. \tag{4.67}$$

From the theory of Lie groups, we know that the exponential map generates group members from their associated elements of the Lie algebra. Conversely, we obtain an element of the Lie algebra by differentiating a curve of group elements at $t=0$. This procedure is similar to the generation of a tangent vector by the differentiation of a curve through a point manifold. Making use of this fact, we are going to take the derivative of (4.66) at $t=0$, which is the neighborhood of the identity element of the Lorentz group. This way, the isometric requirement for group members Λ of SO(1,3) turns into a restriction for the elements M of the Lie algebra $\mathfrak{so}(1,3)$:

$$\begin{aligned} \frac{d}{dt}\big[\Lambda(t)\, g\, \Lambda^T(t)\big]\Big|_{t=0} \;&=\; \frac{d}{dt}\,\Lambda(t)\cdot g\,\Lambda^T(t)\Big|_{t=0} +\; \Lambda(t)\, g\cdot \frac{d}{dt}\,\Lambda^T(t)\Big|_{t=0} \\ &=\; Mg\,\mathbb{1} \;+\; \mathbb{1}\, gM^T \;=\; 0 \qquad (4.68) \\ &\Rightarrow\; (Mg)^T = -\,Mg. \qquad (4.69) \end{aligned}$$

In the last step, we made use of the symmetry $g^T = g$ of the metric tensor. Although equation (4.69) doesn't make M itself antisymmetric, it does so for the product Mg. This "extended antisymmetry" turns into a true antisymmetry in case of a Euclidean metric $g = \mathbb{1}$ as e.g. for the Lie algebra $\mathfrak{so}(3)$. The properties (4.66) and (4.69) fit well with the characteristics of the matrices $\Lambda \in$ SO(1,3) and $M \in \mathfrak{so}(1,3)$:

– The concatenation of Lorentz transformations (or matrix multiplication) is closed with respect to the isometric requirement because of

$$\Lambda_1\Lambda_2\, g\,(\Lambda_1\Lambda_2)^T \,=\, \Lambda_1\Lambda_2\, g\,\Lambda_2^T\Lambda_1^T \,=\, \Lambda_1\, g\,\Lambda_1^T \,=\, g\,. \tag{4.70}$$

  On the other hand, a linear combination of Lorentz transformations generally doesn't meet the isometric requirement and therefore doesn't lead to another Lorentz transformation. In other words, Lorentz transformations form a group with respect to matrix multiplication, but they don't form a vector space regarding addition.

– It is easy to see that the extended antisymmetry (4.69) of the matrices M also holds for linear combinations. However, it is not preserved in a product of two matrices M_1 and M_2:

$$\begin{aligned}(M_1 M_2\, g)^T &= (M_1\, g g^{-1} M_2\, g)^T = (M_2\, g)^T g^{-1} (M_1\, g)^T \\ &= M_2\, g g^{-1} M_1\, g = M_2 M_1\, g \neq -M_1 M_2\, g\,.\end{aligned} \tag{4.71}$$

In the second step, we applied (4.69) as well as the symmetry of the metric tensor. The Lie bracket, which in our case takes the form of a commutator, is the characteristic operation of this Lie algebra. As the algebra operation of $\mathfrak{so}(1,3)$, it should be closed within the class of extended antisymmetric matrices M:

$$\begin{aligned}\big([M_1, M_2]\, g\big)^T &= \big(M_1 M_2\, g\big)^T - \big(M_2 M_1\, g\big)^T \\ &= M_2 M_1\, g - M_1 M_2\, g = -[M_1, M_2]\, g\,.\end{aligned} \tag{4.72}$$

Relation (4.71) was applied in the evaluation of the transposed products. The above–mentioned properties make the extended antisymmetric matrices a vector space and an algebra with respect to the Lie bracket, but regarding matrix multiplication, they don't form a group.

As the following step, we will construct a basis for the Lie algebra $\mathfrak{so}(1,3)$. Instead of doing so directly, we will first look for a basis of the antisymmetric 4×4–matrices Mg. The method presented here gives us a basis in the six–dimensional space of antisymmetric 4×4–matrices, which will be an obvious extension of the basis for the three–dimensional Lie algebra $\mathfrak{so}(3)$, see [3]:

$$\begin{aligned}
M_1\, g &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \qquad M_4\, g &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
M_2\, g &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \qquad M_5\, g &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
M_3\, g &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \qquad M_6\, g &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{4.73}$$

We obtain the basis matrices M_i themselves by multiplying g^{-1} from the right, which gives us three antisymmetric matrices and three symmetric ones. At this point, we also would like to change the designation:

$$
\begin{aligned}
P_x &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & J_x &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
P_y &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & J_y &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
P_z &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & J_z &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{4.74}
$$

The three matrices P_x, P_y and P_z obviously can be considered as additional generators of the rotation group SO(3) [3] which extend the latter into a "Zero" dimension. The generators J_x, J_y and J_z of the rotation group don't act on this additional dimension, whereas P_x, P_y and P_z involve this zero– or time dimension in particular. This allows us to interpret them as generators of special Lorentz transformations along the coordinate axes. All six base matrices are generated from their corresponding elements of the Lorentz– or the rotation group through differentiation, cf. (4.63)–(4.65) and [3]:

$$
P_x = -\frac{d}{dt}\,\Lambda_S^{(1)}(t)\Big|_{t=0} \qquad P_y = -\frac{d}{dt}\,\Lambda_S^{(2)}(t)\Big|_{t=0} \qquad P_z = -\frac{d}{dt}\,\Lambda_S^{(3)}(t)\Big|_{t=0} \tag{4.75}
$$

$$
J_x = \frac{d}{dt}\,\mathbf{R}(\underline{e}_x, t)\Big|_{t=0} \qquad J_y = \frac{d}{dt}\,\mathbf{R}(\underline{e}_y, t)\Big|_{t=0} \qquad J_z = \frac{d}{dt}\,\mathbf{R}(\underline{e}_z, t)\Big|_{t=0} \tag{4.76}
$$

The minus sign in (4.75) originates from the conventions regarding contravariant components that we established in section 2.2. The physical context of the parameter t is either the rapidity χ for Lorentz boosts or the rotation angle θ in case of rotations.

Our study of quaternions and Pauli matrices has already shown that a multiplication table helps to summarize the relations between their members in a concise manner. In the context of a Lie algebra, however, we are not talking about the matrix multiplication of its elements B_i. We are talking about the algebra operation, which is the Lie bracket $[B_i, B_j]$. The outcome of the latter defines so–called "Structure constants" ${C_{ij}}^k$ as follows:

$$
[B_i, B_j] \;=\; {C_{ij}}^k B_k\,. \tag{4.77}
$$

In (4.77) the generators P_i and J_i were subsumed into general operators B_i. Explicit calculation reveals that the structure constants of the Lie algebra $\mathfrak{so}(1,3)$ arrange in several groups. For the boost generators we have

$$\left.\begin{array}{rcl}[P_x,P_y] &=& P_z\\ [P_y,P_z] &=& P_x\\ [P_z,P_x] &=& P_y\end{array}\right\} \Rightarrow\ {C_{ij}}^k \;=\; \begin{cases} 1 & i\,j\,k \text{ even permutation of } x\,y\,z\\ -1 & i\,j\,k \text{ odd permutation of } x\,y\,z\\ 0 & \text{else.}\end{cases} \tag{4.78}$$

For the rotation generators it is

$$\left.\begin{array}{rcl}[J_x,J_y] &=& J_z\\ [J_y,J_z] &=& J_x\\ [J_z,J_x] &=& J_y\end{array}\right\} \Rightarrow\ {C_{ij}}^k \;=\; \begin{cases} 1 & i\,j\,k \text{ even permutation of } x\,y\,z\\ -1 & i\,j\,k \text{ odd permutation of } x\,y\,z\\ 0 & \text{else.}\end{cases} \tag{4.79}$$

Mixed commutators which involve both types of generators look like

$$\left.\begin{array}{rcccl}[P_x,J_y] &=& [J_x,P_y] &=& P_z\\ [P_y,J_z] &=& [J_y,P_z] &=& P_x\\ [P_z,J_x] &=& [J_z,P_x] &=& P_y\end{array}\right\} \Rightarrow\ {C_{ij}}^k \;=\; \begin{cases} 1 & i\,j\,k \text{ even pm.}(x\,y\,z)\\ -1 & i\,j\,k \text{ odd pm.}(x\,y\,z)\\ 0 & \text{else.}\end{cases} \tag{4.80}$$

In section 1.2, we have found that the rotation group SO(3) forms a subgroup within the Lorentz group SO(1,3). This becomes visible in the basis of the Lie algebra $\mathfrak{so}(1,3)$, which directly expands that of the Lie algebra $\mathfrak{so}(3)$ into a "Zero"– or time dimension. For the rest of this section we want to concentrate on Lorentz boosts without rotations and therefore restrict the Lie algebra $\mathfrak{so}(1,3)$ to the subspace which is spanned by the P_i alone.

The generators P_x, P_y and P_z of boosts along the coordinate axes allow for an angle–axis like construction of a general boost. In order to achieve this, we parameterize the Lie algebra $\mathfrak{so}(1,3)$ appropriately:

$$t\,\mathbf{n} \;:=\; t\,(n^x P_x + n^y P_y + n^z P_z) \qquad \text{with} \qquad (n^x)^2 + (n^y)^2 + (n^z)^2 \;=\; 1\,. \tag{4.81}$$

The quantity $\mathbf{n}$ which we introduced above represents a direction vector $\underline{n}$ within the Lie algebra $\mathfrak{so}(1,3)$, while the parameter t stands for the rapidity. Applying the exponential map, we generate the associated Lorentz transformations $\Lambda(\underline{n},t)$ from the elements (4.81) of the Lie algebra:

$$\Lambda(\underline{n},t) \;=\; \mathrm{e}^{t\,\mathbf{n}} \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\,(t\,\mathbf{n})^k\;. \tag{4.82}$$

We have to emphasize again that the generators P_i don't commute (4.78), which means that we can't apply elementary exponential laws. A decomposition according to Euler's formula (4.25) doesn't work either, as $\mathbf{n}^2 = \pm\mathbb{1}$ isn't valid here. Instead, we have to evaluate the power series expansion (4.82) itself, which is the very definition of the exponential function. This requires additional information on the powers of the direction vector $\mathbf{n}$. At first we aggregate the components of $\mathbf{n}$ into a matrix:

$$\mathbf{n} = \begin{pmatrix} 0 & n^x & n^y & n^z \\ n^x & 0 & 0 & 0 \\ n^y & 0 & 0 & 0 \\ n^z & 0 & 0 & 0 \end{pmatrix} . \tag{4.83}$$

Then we determine the powers of $\mathbf{n}$ by explicit calculation:

$$\mathbf{n}^0 = \mathbb{1} \tag{4.84}$$

$$\mathbf{n}^1 = \begin{pmatrix} 0 & n^x & n^y & n^z \\ n^x & 0 & 0 & 0 \\ n^y & 0 & 0 & 0 \\ n^z & 0 & 0 & 0 \end{pmatrix} \tag{4.85}$$

$$\mathbf{n}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n^{x2} & n^x n^y & n^x n^z \\ 0 & n^y n^x & n^{y2} & n^y n^z \\ 0 & n^z n^x & n^z n^y & n^{z2} \end{pmatrix} \tag{4.86}$$

$$\mathbf{n}^3 = \begin{pmatrix} 0 & n^x & n^y & n^z \\ n^x & 0 & 0 & 0 \\ n^y & 0 & 0 & 0 \\ n^z & 0 & 0 & 0 \end{pmatrix} = \mathbf{n} . \tag{4.87}$$

Our calculations took into account that the direction vector is normalized, which means $(n^x)^2 + (n^y)^2 + (n^z)^2 = 1$. At first glance, the relation $\mathbf{n}^3 = \mathbf{n}$ without also having $\mathbf{n}^2 = \mathbb{1}$ might appear strange, but $\mathbf{n}$ isn't invertible as its determinant vanishes. With (4.84)–(4.87) we can write down all powers of the direction vector

$$\begin{aligned} \mathbf{n}^0 &= \mathbb{1} \\ \mathbf{n}^{2k+1} &= \mathbf{n} \qquad 0 \leq k < \infty \\ \mathbf{n}^{2k} &= \mathbf{n}^2 \qquad 1 \leq k < \infty , \end{aligned} \tag{4.88}$$

and the exponential map (4.82) of the Lorentz boost decomposes into three parts:

$$\Lambda(\underline{n}, t) = \mathbb{1} + \sum_0^\infty \frac{t^{2k+1}}{(2k+1)!} \cdot \mathbf{n} + \sum_1^\infty \frac{t^{2k}}{(2k)!} \cdot \mathbf{n}^2 . \tag{4.89}$$

As the remaining sums no longer involve any generators, they simply represent the power series expansion of hyperbolic functions:

$$\sinh(t) = \sum_0^\infty \frac{t^{2k+1}}{(2k+1)!} \tag{4.90}$$

$$\cosh(t) = \sum_0^\infty \frac{t^{2k}}{(2k)!} . \tag{4.91}$$

With the formulas above, we can finally rewrite the Lorentz boost (4.89) as

$$\Lambda(\underline{n}, t) \;=\; \mathbb{1} \,+\, \sinh(t) \cdot \mathbf{n} \,+\, (\cosh(t) - 1) \cdot \mathbf{n}^2 \,. \tag{4.92}$$

In order to check how this form of a Lorentz boost is compatible with the expression (4.61) that we have already found, we will replace the hyperbolic functions of the rapidity χ with $\gamma-$ and β factors:

$$\Lambda(\underline{n}, t) \;=\; \mathbb{1} \,-\, \gamma\beta \cdot \mathbf{n} \,+\, (\gamma - 1) \cdot \mathbf{n}^2 \,. \tag{4.93}$$

According to (4.59), an additional minus sign was added because we don't want a transformation of base vectors but for components instead. Here the terms of (4.93) are explicitly written down:

$$\gamma\beta \cdot \mathbf{n} \,=\, \begin{pmatrix} 0 & \gamma\beta\, n^x & \gamma\beta\, n^y & \gamma\beta\, n^z \\ \gamma\beta\, n^x & 0 & 0 & 0 \\ \gamma\beta\, n^y & 0 & 0 & 0 \\ \gamma\beta\, n^z & 0 & 0 & 0 \end{pmatrix} \tag{4.94}$$

$$(\gamma - 1) \cdot \mathbf{n}^2 \,=\, \begin{pmatrix} \gamma - 1 & 0 & 0 & 0 \\ 0 & (\gamma-1)\, n^{x2} & (\gamma-1)\, n^x n^y & (\gamma-1)\, n^x n^z \\ 0 & (\gamma-1)\, n^x n^y & (\gamma-1)\, n^{y2} & (\gamma-1)\, n^y n^z \\ 0 & (\gamma-1)\, n^x n^z & (\gamma-1)\, n^y n^z & (\gamma-1)\, n^{z2} \end{pmatrix}. \tag{4.95}$$

Aggregating all parts of (4.93) into a single matrix, the Lorentz boost finally reads

$$\Lambda(\underline{n}, t) = \begin{pmatrix} \gamma & -\gamma\beta\, n^x & -\gamma\beta\, n^y & -\gamma\beta\, n^z \\ -\gamma\beta\, n^x & (\gamma-1)\, n^{x2} + 1 & (\gamma-1)\, n^x n^y & (\gamma-1)\, n^x n^z \\ -\gamma\beta\, n^y & (\gamma-1)\, n^x n^y & (\gamma-1)\, n^{y2} + 1 & (\gamma-1)\, n^y n^z \\ -\gamma\beta\, n^z & (\gamma-1)\, n^x n^z & (\gamma-1)\, n^y n^z & (\gamma-1)\, n^{z2} + 1 \end{pmatrix}. \tag{4.96}$$

The above result demonstrates that the Lie algebra $\mathfrak{so}(1,3)$ leads us to the same Lorentz boost as the Pauli matrix representation, which had been inspired by quaternions and was introduced at the beginning of this chapter.

Finally, we are going to study the effect of the Lorentz boost (4.92) on a world vector

$$x \,=\, \big(ct,\, \underline{x}\big)^{T} \,=\, \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{4.97}$$

in detail and rewrite it with elementary vector operations. In (4.97) we introduced some double designations which we have to straighten. We leave x as the name for the world vector itself and a component of the position vector $\underline{x}$, because both can be easily distinguished based on the underlying context. We will, however, rename the parameter t of the Lorentz boost as χ in order to avoid confusion between the rapidity and the time t. First we will examine how the powers $\mathbf{n}$ and $\mathbf{n}^2$ of the direction vector operate on the world vector x:

$$\mathbf{n}\,x = \begin{pmatrix} n^x x + n^y y + n^z z \\ ct\, n^x \\ ct\, n^y \\ ct\, n^z \end{pmatrix} = \big((\underline{n}, \underline{x}),\ ct\,\underline{n}\,\big)^T \tag{4.98}$$

$$\mathbf{n}^2\,x = \begin{pmatrix} ct \\ (n^x x + n^y y + n^z z)\, n_x \\ (n^x x + n^y y + n^z z)\, n_y \\ (n^x x + n^y y + n^z z)\, n_z \end{pmatrix} = \big(ct,\ (\underline{n}, \underline{x})\,\underline{n}\,\big)^T. \tag{4.99}$$

Plugging these results into (4.92) gives us the time and space coordinates of x after the boost:

$$\begin{aligned} \big[\Lambda(\underline{n},\chi)\,x\,\big]^{(ct)} &= \ \cosh(\chi)\,ct\, +\, \sinh(\chi)\,(\underline{n}, \underline{x}) \\ \big[\Lambda(\underline{n},\chi)\,x\,\big]^{(\underline{x})} &= \big[\sinh(\chi)\,ct\, +\, \cosh(\chi)\,(\underline{n}, \underline{x})\,\big]\,\underline{n}\, +\, \underline{x} - (\underline{n}, \underline{x})\,\underline{n}\,. \end{aligned} \tag{4.100}$$

This form of the Lorentz transformation very much resembles the rotation formula of Olinde Rodrigues. For a better interpretation of the expressions above, we split the vector $\underline{x}$ into parallel and orthogonal components with respect to the direction $\underline{n}$ of the boost:

$$\begin{aligned} \underline{x}_{\,\parallel} &= (\underline{n}, \underline{x})\,\underline{n} =: x_{\,\parallel}\,\underline{n} \\ \underline{x}_{\,\perp} &= \underline{x} - (\underline{n}, \underline{x})\,\underline{n} = \underline{x} - x_{\,\parallel}\,\underline{n}\,. \end{aligned} \tag{4.101}$$

When we apply these definitions to (4.100), the transformed world vector changes into

$$\begin{aligned} \big[\Lambda(\underline{n},\chi)\,x\,\big]^{(ct)} &= \ \cosh(\chi)\,ct\, +\, \sinh(\chi)\,x_{\,\parallel} \\ \big[\Lambda(\underline{n},\chi)\,x\,\big]^{(\underline{x})} &= \big[\sinh(\chi)\,ct\, +\, \cosh(\chi)\,x_{\,\parallel}\,\big]\,\underline{n}\, +\, \underline{x}_{\,\perp}\,. \end{aligned} \tag{4.102}$$

Here it becomes apparent that a general boost behaves like a special Lorentz boost along the direction vector $\underline{n}$ and leaves the orthogonal component of $\underline{x}$ untouched. Although this result might not exactly come as a surprise and the general Lorentz boost could have been constructed like this anyway, (4.102) is at least a helpful check of our calculations. The juxtaposition with Rodrigues' rotation formula reveals how similar both expressions are [3]:

$$\begin{aligned} \big[\Lambda(\underline{n},\chi)\,x\,\big]^{(ct)} &= \ \cosh(\chi)\,ct\, +\, \sinh(\chi)\,x_{\,\parallel} \\ \big[\Lambda(\underline{n},\chi)\,x\,\big]^{(\underline{x})} &= \big[\sinh(\chi)\,ct\, +\, \cosh(\chi)\,x_{\,\parallel}\,\big]\,\underline{n}\, +\, \underline{x}_{\,\perp} \\ \mathbf{R}(\underline{n},\theta)\,\underline{x} &= \big[\cos(\theta)\,\underline{n}_1\, -\, \sin(\theta)\,\underline{n}_2\big]x_\perp\, +\, \underline{x}_{\,\parallel}\,. \end{aligned} \tag{4.103}$$

In the Rodrigues formula, the vectors $\underline{n}_1$ and $\underline{n}_2$ form an orthonormal base within the plane of rotation with $\underline{n}_1$ pointing in the direction of $\underline{x}_{\,\perp}$. Of course, the parameters $\underline{n}$ and χ or θ have varying physical interpretations, $\underline{n}$ stands for the direction of either the Lorentz boost or the axis of rotation, χ is the rapidity and θ the rotation angle.

5 The Thomas–Wigner Rotation

In section 4.4, our study on adding velocities has revealed that the concatenation of two Lorentz boosts doesn't necessarily lead to another pure boost. The result might rather include a unitary component which then can be split off by polar decomposition. This unitary component is called the "Thomas–Wigner Rotation", which we are going to have a deeper look into in this chapter. For this purpose, we will compare the product of two boosts with the polar decomposition of one boost and try to extract the rotation parameters of the latter.

5.1 Unified Representation of Boosts and Rotations

For the upcoming calculations we have to find a common notation that applies to both Lorentz transformations and rotations. We have arlready encountered three promising candidates in the last chapter: Quaternionic numbers as well as the matrix groups SL(2,$\mathbb{C}$) and SO(1,3). We choose the group SL(2,$\mathbb{C}$) because the Pauli matrices of section 4.1 provide us with a handy way to elegantly manipulate elements of this matrix group algorithmically. The group SL(2,$\mathbb{C}$) represents the entire proper Lorentz group including those special cases:

– Hermitian matrices: They stand for Lorentz boosts and don't form a group by themselves.

– Special unitary matrices: They represent rotations and form the subgroup SU(2).

We can write down a Lorentz boost explicitly with the generalized Euler formula (4.25):

$$\begin{aligned} L_B &= \mathrm{e}^{-(\chi/2)\mathbf{n}} = \cosh(\chi/2)\,\boldsymbol{\sigma}_0 - \sinh(\chi/2)\,\mathbf{n} \\ \mathbf{n} &= n^1\boldsymbol{\sigma}_1 + n^2\boldsymbol{\sigma}_2 + n^3\boldsymbol{\sigma}_3 \end{aligned} \quad . \tag{5.1}$$

RED ALERT!! German text ahead...

Dieser Ausdruck ähnelt sehr der Darstellung einer Rotation als Quaternion [3]:

$$\begin{aligned} U &= \mathrm{e}^{(\theta/2)\mathbf{n}} = \cos(\theta/2) + \sin(\theta/2)\,\mathbf{n} \\ \mathbf{n} &= n^1\,\mathbf{i} + n^2\,\mathbf{j} + n^3\mathbf{k} \end{aligned} \quad . \tag{5.2}$$

Mit Hilfe der Paulimatrizen kann auch das Rotationsquaternion U als komplexe 2×2–Matrix geschrieben werden:

$$\mathbb{1} = \boldsymbol{\sigma}_0 \qquad \mathbf{i} = -i\boldsymbol{\sigma}_1 \qquad \mathbf{j} = -i\boldsymbol{\sigma}_2 \qquad \mathbf{k} = -i\boldsymbol{\sigma}_3 \tag{5.3}$$

Die Matrizen $\boldsymbol{\sigma}_\mu$ sind die Paulimatrizen $\boldsymbol{\sigma}_i$ ergänzt durch die Einheitsmatrix $\boldsymbol{\sigma}_0$. Damit agieren Lorentz–Boosts und Rotationen im gleichen Darstellungsraum:

$$L_B(\underline{n}_{\mathrm{B}}, \chi) = \cosh(\chi/2)\, \boldsymbol{\sigma}_0 \,-\, \sinh(\chi/2)\, \mathbf{n}_{\mathrm{B}} \tag{5.4}$$

$$U(\underline{n}_{\mathrm{R}}, \theta) = \cos(\theta/2)\, \boldsymbol{\sigma}_0 \,-\, i \sin(\theta/2)\, \mathbf{n}_{\mathrm{R}}. \tag{5.5}$$

Hierbei ist $\mathbf{n}_{\mathrm{B}}$ (bzw. $\underline{n}_{\mathrm{B}}$) der Richtungsvektor der Geschwindigkeit des Lorentz–Boosts, und $\mathbf{n}_{\mathrm{R}}$ (bzw. $\underline{n}_{\mathrm{R}}$) ist der Normalenvektor der Drehachse der Rotation. Wie zu Beginn des Kapitels angedeutet haben die Matrizen $L_B(\underline{n}_{\mathrm{B}}, \chi)$ und $U(\underline{n}_{\mathrm{R}}, \theta)$ trotz ihrer ähnlichen Form unterschiedliche Eigenschaften:

$L_B(\underline{n}_{\mathrm{B}}, \chi)$: Hermitesch, nur Element von SL(2,$\mathbb{C}$)

$U(\underline{n}_{\mathrm{R}}, \theta)$: Unitär, Element von SL(2,$\mathbb{C}$) und von SU(2).

5.2 Polar Decomposition of Two Consecutive Boosts

Die Zerlegung einer allgemeinen Lorentztransformation L in ihren reinen Anteil L_B und eine Rotation U kann wie in (1.8) beschrieben auf zweierlei Weise geschehen:

$$\begin{aligned} L \;&= U L_B \,=\, L'_B U \\ L'_B &= U L_B U^\dagger \end{aligned} \quad . \tag{5.6}$$

Bevor wir die Polarzerlegung zweier Lorentz–Boosts im Detail besprechen, sollten wir verstehen, wie sich diese beiden Zerlegungsmethoden voneinander unterscheiden. Zu diesem Zweck setzen drücken wir L_B durch (5.1) aus und erhalten:

$$\begin{aligned} L'_B &= U \big(\cosh(\chi/2)\, \boldsymbol{\sigma}_0 - \sinh(\chi/2)\, \mathbf{n}\big)\, U^\dagger \\ &= \cosh(\chi/2)\, \boldsymbol{\sigma}_0 - \sinh(\chi/2)\, U \mathbf{n}\, U^\dagger \quad . \\ &=: \cosh(\chi/2)\, \boldsymbol{\sigma}_0 - \sinh(\chi/2)\, \mathbf{n}' \end{aligned} \tag{5.7}$$

Die der Rotation U folgenden bzw. vorausgehenden Boosts L'_B und L_B besitzen also dieselbe Rapidität χ. Die Richtungen $\mathbf{n}'$ bzw. $\mathbf{n}$ der beiden Transformationen sind jedoch verschieden:

$$\mathbf{n}' \,=\, U \mathbf{n}\, U^\dagger. \tag{5.8}$$

Dieser Ausdruck beschreibt die Drehung des dem Vektor $\underline{n}$ zugeordneten $(\frac{1}{2}, \frac{1}{2})$–Spinors $\mathbf{n}$ durch den unitären Operator U. Ist also die Zerlegung einer allgemeinen Lorentztransformation L in L_B und U bekannt, läßt sich L'_B einfach durch Drehung des Normalenvektors $\underline{n}$ ermitteln:

$$L'_B \,=\, L_B(U \mathbf{n}\, U^\dagger, \chi). \tag{5.9}$$

Für den Rest dieses Kapitels beschränken wir uns auf den Fall

$$L = UL_B \tag{5.10}$$

in welchem der Boost der Rotation vorausgeht und wenden uns nun der Komposition zweier Boosttransformationen zu.

Das Produkt zweier Lorentz–Boosts bestimmen wir mit Hilfe der Rechenregeln für die Paulimatrizen (4.8) und deren Multiplikationstabelle (4.9):

$$\begin{aligned}
&\boldsymbol{\sigma}_1{}^2 = \boldsymbol{\sigma}_2{}^2 = \boldsymbol{\sigma}_3{}^2 = -i\,\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_3 = \mathbb{1} && (5.11)\\
&\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_3 &&\\
&\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_3 = -\boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_1 && (5.12)\\
&\boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_2. &&
\end{aligned}$$

Unter Ausnutzung dieser Beziehungen lautet das Produkt zweier Boosts

$$\begin{aligned}
&L_B(\underline{n}_2,\chi_2)\,L_B(\underline{n}_1,\chi_1) = \\
&\quad = \big(\cosh(\chi_2/2)\,\boldsymbol{\sigma}_0 - \sinh(\chi_2/2)\,\mathbf{n}_2\big)\big(\cosh(\chi_1/2)\,\boldsymbol{\sigma}_0 - \sinh(\chi_1/2)\,\mathbf{n}_1\big). \qquad (5.13)\\
&\quad = \mathrm{ch}_1\,\mathrm{ch}_2\,\boldsymbol{\sigma}_0 - \mathrm{sh}_1\,\mathrm{ch}_2\,\mathbf{n}_1 - \mathrm{sh}_2\,\mathrm{ch}_1\,\mathbf{n}_2 + \mathrm{sh}_1\,\mathrm{sh}_2\,\mathbf{n}_2\mathbf{n}_1
\end{aligned}$$

Um die Übersicht über die Terme in (5.13) zu behalten haben wir naheliegende Abkürzungen der Gestalt $\mathrm{ch}_1 = \cosh(\chi_1/2)$ usw. eingeführt. Der Ausdruck $\mathbf{n}_2\mathbf{n}_1$ hat die Struktur eines Clifford– Produktes, welches in die Summe eines inneren und eines äußeren Produktes zerfällt:

$$\begin{aligned}
\mathbf{n}_2\mathbf{n}_1 &= \big(n_2^i\,\boldsymbol{\sigma}_i\big)\big(n_1^j\,\boldsymbol{\sigma}_j\big)\\
&= \sum_{i=1}^{3} n_2^i\,n_1^i\,\boldsymbol{\sigma}_0 + i\left(n_2^1\,n_1^2 - n_2^2\,n_1^1\right)\boldsymbol{\sigma}_3 + \cdots\\
&\qquad\cdots + i\left(n_2^3\,n_1^1 - n_2^1\,n_1^3\right)\boldsymbol{\sigma}_2 + i\left(n_2^2\,n_1^3 - n_2^3\,n_1^2\right)\boldsymbol{\sigma}_1 \qquad (5.14)\\
&= (\underline{n}_2,\underline{n}_1)\,\boldsymbol{\sigma}_0 + i\,\mathbf{n}_2\wedge\mathbf{n}_1
\end{aligned}$$

Das äußere Produkt $\mathbf{n}_2\wedge\mathbf{n}_1$ faßt die Terme mit den Komponenten von $\underline{n}_2\times\underline{n}_1$ in einer kompakten Schreibweise zusammen. Setzen wir (5.14) in (5.13) ein, erhalten wir:

$$\begin{aligned}
L_B(\underline{n}_2,\chi_2)\,L_B(\underline{n}_1,\chi_1) &= \big(\mathrm{ch}_1\,\mathrm{ch}_2 + \mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1,\underline{n}_2)\big)\,\boldsymbol{\sigma}_0 - \cdots\\
&\qquad\cdots - \mathrm{sh}_1\,\mathrm{ch}_2\,\mathbf{n}_1 - \mathrm{sh}_2\,\mathrm{ch}_1\,\mathbf{n}_2 + i\,\mathrm{sh}_1\,\mathrm{sh}_2\,\mathbf{n}_2\wedge\mathbf{n}_1 .
\end{aligned} \tag{5.15}$$

Andererseits ergibt die Polarzerlegung der resultierenden Lorentztransformation

$$\begin{aligned}
&U(\underline{n}_{\mathrm{R}},\theta)\,L_B(\underline{n}_{\mathrm{B}},\chi) = \\
&\quad = \big(\cos(\theta/2)\,\boldsymbol{\sigma}_0 - i\sin(\theta/2)\,\mathbf{n}_{\mathrm{R}}\big)\big(\cosh(\chi/2)\,\boldsymbol{\sigma}_0 - \sinh(\chi/2)\,\mathbf{n}_{\mathrm{B}}\big), \qquad (5.16)\\
&\quad = \cos\mathrm{ch}\,\boldsymbol{\sigma}_0 - i\sin\mathrm{ch}\,\mathbf{n}_{\mathrm{R}} - \cos\mathrm{sh}\,\mathbf{n}_{\mathrm{B}} + i\sin\mathrm{sh}\,\mathbf{n}_{\mathrm{R}}\mathbf{n}_{\mathrm{B}}
\end{aligned}$$

worin wir dieselbe Art von Abkürzungen eingeführt haben wie in (5.13). Das Clifford–Produkt der Achsen $\mathbf{n}_{\mathrm{R}}$ und $\mathbf{n}_{\mathrm{B}}$ lautet (vgl. (5.14)):

$$\mathbf{n}_{\mathrm{R}}\mathbf{n}_{\mathrm{B}} = (\underline{n}_{\mathrm{R}}, \underline{n}_{\mathrm{B}})\,\boldsymbol{\sigma}_0 + i\,\mathbf{n}_{\mathrm{R}}\wedge\mathbf{n}_{\mathrm{B}}. \tag{5.17}$$

Nach Einsetzen des Clifford– Produktes (5.17) in den Ausdruck (5.16) erhalten wir

$$\begin{aligned} U(\underline{n}_{\mathrm{R}}, \theta)\, L_B(\underline{n}_{\mathrm{B}}, \chi) = \big(\cos\mathrm{ch} + i \sin\mathrm{sh}\,(\underline{n}_{\mathrm{R}}, \underline{n}_{\mathrm{B}})\big)\,\boldsymbol{\sigma}_0 - \cdots \\ \cdots - \cos\mathrm{sh}\,\mathbf{n}_{\mathrm{B}} - \sin\mathrm{sh}\,\mathbf{n}_{\mathrm{R}}\wedge\mathbf{n}_{\mathrm{B}} - i\sin\mathrm{ch}\,\mathbf{n}_{\mathrm{R}} \end{aligned}. \tag{5.18}$$

Der Polarzerlegungssatz besagt, daß man (5.18) mit der Darstellung (5.15) des Produktes zweier Lorentz–Boosts gleichsetzen kann. Beide Gleichungen enthalten einen Skalarteil ($\boldsymbol{\sigma}_0$) und einen Vektorteil, die jeweils beide in einen Realteil und einen Imaginärteil zerfallen. Durch den Vergleich dieser vier Anteile ergeben sich vier Gleichungen in den Größen θ, χ, $\underline{n}_{\mathrm{R}}$ und $\underline{n}_{\mathrm{B}}$:

$$\mathrm{ch}_1\,\mathrm{ch}_2 + \mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1, \underline{n}_2) = \cos\mathrm{ch} \tag{5.19}$$

$$\sin\mathrm{sh}\,(\underline{n}_{\mathrm{R}}, \underline{n}_{\mathrm{B}}) = 0 \tag{5.20}$$

$$\mathrm{sh}_1\,\mathrm{ch}_2\,\underline{n}_1 + \mathrm{sh}_2\,\mathrm{ch}_1\,\underline{n}_2 = \cos\mathrm{sh}\,\underline{n}_{\mathrm{B}} + \sin\mathrm{sh}\,\underline{n}_{\mathrm{R}}\times\underline{n}_{\mathrm{B}} \tag{5.21}$$

$$\mathrm{sh}_1\,\mathrm{sh}_2\,\underline{n}_1\times\underline{n}_2 = \sin\mathrm{ch}\,\underline{n}_{\mathrm{R}}. \tag{5.22}$$

Da die obenstehenden Gleichungen gelten komponentenweise, weswegen wir von der Darstellung in Paulimatrizen zur Vektorschreibweise übergehen konnten.

5.3 Plausibility of the Equation System

Es ist gewiß von Vorteil, vor den ersten Lösungsversuchen die Plausibilität des Gleichungssystems (5.19) – (5.22) zu diskutieren und zu überprüfen, ob bereits bekannte Spezialfälle durch diese Gleichungen reproduziert werden. Zunächst jedoch beschäftigen wir uns mit zwei Eigenschaften, die sofort ins Auge fallen:

- Gleichung (5.20) besagt, daß wegen $(\underline{n}_{\mathrm{R}}, \underline{n}_{\mathrm{B}}) = 0$ die Drehachse $\underline{n}_{\mathrm{R}}$ senkrecht zur Geschwindigkeitsrichtung $\underline{n}_{\mathrm{B}}$ der resultierenden Lorentztransformation steht. In den Fällen $\sin(\theta/2) = 0$ und $\sinh(\chi/2) = 0$ bleibt das obenstehende Skalarprodukt zwar unbestimmt, allerdings ist für $\theta = 0$ die Lage der Drehachse und für $\chi = 0$ die Lage des Geschwindigkeitsvektors irrelevant.
- Gleichung (5.22) besagt, daß wegen $\underline{n}_1\times\underline{n}_2 \sim \underline{n}_{\mathrm{R}}$ die Drehachse $\underline{n}_{\mathrm{R}}$ senkrecht zu den Geschwindigkeitsrichtungen $\underline{n}_1$ und $\underline{n}_2$ der kombinierten Boosts steht. In den Fällen $\sinh(\chi_1/2) = 0$, $\sinh(\chi_2/2) = 0$ und $\sin(\theta/2) = 0$ bleibt die relative Lage des Kreuzproduktes zur Drehachse allerdings unbestimmt. Die

Frage nach der relativen Orientierung der Drechachse in Bezug zu den Boostgeschwindigkeiten verliert jedoch ihren Sinn, wenn wegen $\chi_1 = 0$ bzw. $\chi_2 = 0$ jeweils ein Geschwindigkeitsvektor verschwindet oder wenn wegen $\theta = 0$ keine Drehung stattfindet.

Insgesamt steht also die Drehachse $\underline{n}_{\mathrm{R}}$ der Thomas–Wigner Rotation sowohl senkrecht zu den Ausgangsgeschwindigkeiten als auch senkrecht zur Endgeschwindigkeit nach der Addition, weswegen die Vektoren $\underline{n}_1$, $\underline{n}_2$ und $\underline{n}_{\mathrm{B}}$ in einer Ebene liegen müssen. Die Rotation findet ebenfalls in dieser Ebene statt, womit die hierzu senkrechte Dimension vom Prozeß der beiden Lorentztransformationen gänzlich unberührt bleibt. Das war zu erwarten, weil die Symmetrie des Vorgangs diesen auf die von $\underline{n}_1$ und $\underline{n}_2$ aufgespannte Ebene beschränkt.

Als nächstes überprüfen wir die Gleichungen (5.19) – (5.22) anhand einiger Sonderfälle, deren Ergebnis wir bereits kennen:

$v_1 = 0$ *bzw.* $v_2 = 0$:

Bei verschwindendem v_1 bzw. v_2 trägt jeweils nur ein Lorentz–Boost zum Ergebnis bei. In Gleichung (5.22) verschwindet wegen $\chi_1 = 0$ bzw. $\chi_2 = 0$ die linke Seite und legt so den Drehwinkel fest:

$$\sin(\theta/2) = 0 \quad \Rightarrow \quad \theta = 0. \tag{5.23}$$

Mit $\theta = 0$ ist Gleichung (5.20) automatisch erfüllt und liefert keine zusätzlichen Informationen. Es bleiben noch zwei Gleichungen, die wir für $\chi_1 = 0$ hinschreiben:

$$\begin{array}{lrcl} (5.19): & \cosh(\chi_2/2) & = & \cosh(\chi/2) \\ (5.21): & \sinh(\chi_2/2)\,\underline{n}_2 & = & \sinh(\chi/2)\,\underline{n}_{\mathrm{B}} \end{array} . \tag{5.24}$$

Damit sind die Größen χ und $\underline{n}_{\mathrm{B}}$ ebenfalls festgelegt und wir können zusammenfassen:

$$\begin{array}{llll} v_1 = 0: & \theta = 0, & \chi = \chi_2, & \underline{n}_{\mathrm{B}} = \underline{n}_2 \\ v_2 = 0: & \theta = 0, & \chi = \chi_1, & \underline{n}_{\mathrm{B}} = \underline{n}_1 \end{array} . \tag{5.25}$$

Das hier ebenfalls angegebene Resultat für $v_2 = 0$ läßt sich analog zum Fall $v_1 = 0$ berechnen. Für die Lorentztransformationen selbst gilt:

$$\begin{array}{rcl} L_B(\underline{n}_2, \chi_2)\, L_B(\underline{n}_1, 0) & = & L_B(\underline{n}_2, \chi_2) \\ L_B(\underline{n}_2, 0)\, L_B(\underline{n}_1, \chi_1) & = & L_B(\underline{n}_1, \chi_1) \end{array} . \tag{5.26}$$

Dieses Resultat ist korrekt, denn ein Lorentz–Boost mit der Rapidität 0 wirkt wie die Identitätsoperation.

Parallele Geschwindigkeiten:

Besitzen die beiden Lorentz–Boosts $L_B(\underline{n}_1, \chi_1)$ und $L_B(\underline{n}_2, \chi_2)$ parallele Geschwindigkeiten, verschwindet mit $\underline{n}_2 = \underline{n}_1$ das Kreuzprodukt der beiden Richtungsvektoren. Auf diese Weise wird der Drehwinkel θ durch Gleichung (5.22) festgelegt:

$$\sin(\theta/2) = 0 \quad \Rightarrow \quad \theta = 0. \tag{5.27}$$

Mit $\theta = 0$ ist Gleichung (5.20) automatisch erfüllt und liefert keine zusätzlichen Informationen. Es bleiben noch die beiden Gleichungen

$$\begin{aligned} (5.19):& \quad \mathrm{ch}_1\,\mathrm{ch}_2 \;+\; \mathrm{sh}_1\,\mathrm{sh}_2 \;=\; \mathrm{ch} \\ (5.21):& \quad \big[\mathrm{sh}_1\,\mathrm{ch}_2 \;+\; \mathrm{sh}_2\,\mathrm{ch}_1\big]\,\underline{n}_1 \;=\; \mathrm{sh}\,\underline{n}_{\mathrm{B}} \end{aligned}, \tag{5.28}$$

die mit Hilfe der Additionstheoreme für hyperbolische Funktionen umgeformt werden können zu

$$\begin{aligned} \cosh(\chi_2/2 + \chi_1/2) \;&=\; \cosh(\chi/2) \\ \big[\sinh(\chi_2/2 + \chi_1/2)\big]\,\underline{n}_1 \;&=\; \sinh(\chi/2)\,\underline{n}_{\mathrm{B}} \end{aligned}. \tag{5.29}$$

Als Resultat ergibt sich das bereits aus Abschnitt 4.4 bekannte Additionstheorem für parallele Geschwindigkeiten:

$$\underline{n}_2 = \underline{n}_1: \quad \theta = 0, \quad \chi_2 + \chi_1 = \chi, \quad \underline{n}_1 = \underline{n}_{\mathrm{B}}. \tag{5.30}$$

Für die Lorentztransformationen selbst lautet dieses Theorem

$$L_B(\underline{n}_1, \chi_2)\, L_B(\underline{n}_1, \chi_1) \;=\; L_B(\underline{n}_1, \chi_2 + \chi_1). \tag{5.31}$$

Im Fall paralleler Geschwindigkeiten erhalten wir somit ebenfalls ein korrektes Resultat.

Galileischer Grenzfall:

In einem nichtrelativistischem Szenario sind die Geschwindigkeiten v_1, v_2 und v klein gegenüber der Lichtgeschwindigkeit c, wodurch die Rapiditäten sich den zugehörigen Geschwindigkeiten annähern, vgl. die Einführung der Rapidität (2.28) in Kapitel 2:

$$v \ll c: \quad \chi \approx \frac{v}{c}. \tag{5.32}$$

Wir werden nun die Gleichungen (5.19) – (5.22) in erster Ordnung in den Geschwindigkeiten auswerten und schreiben

$$\begin{aligned} \mathrm{ch}_1 &\approx 1 \qquad \mathrm{sh}_1 \approx \frac{1}{2}\frac{v_1}{c} \\ \mathrm{ch}_2 &\approx 1 \qquad \mathrm{sh}_2 \approx \frac{1}{2}\frac{v_2}{c} \\ \mathrm{ch} &\approx 1 \qquad \mathrm{sh} \;\approx \frac{1}{2}\frac{v}{c}. \end{aligned} \tag{5.33}$$

Der Faktor 1/2 ist eine Folge der Halbrapiditäten mit denen die Funktionen ch und sh im letzten Abschnitt definiert wurden. In dieser Näherung verschwindet die linke Seite von Gleichung (5.22) und wir können schließen:

$$\theta = 0 : \quad \cos(\theta/2) = 1, \quad \sin(\theta/2) = 0. \tag{5.34}$$

Die ersten beiden Gleichungen (5.19) und (5.20) degenerieren ohne weiteren Informationsgewinn zu den trivialen Identitäten $1 = 1$ und $0 = 0$. Es bleibt noch die Gleichung (5.21), die in dieser Näherung die Gestalt

$$\frac{1}{2}\frac{v_1}{c}\,\underline{n}_1 \;+\; \frac{1}{2}\frac{v_2}{c}\,\underline{n}_2 \;=\; \frac{1}{2}\frac{v}{c}\,\underline{n}_{\mathrm{B}} \tag{5.35}$$

annimmt. Vereinigen wir Betrag und Richtungsvektor der Geschwindigkeiten zu vektoriellen Größen, erhalten wir mit

$$v \ll c : \quad \theta = 0, \quad \underline{v}_1 + \underline{v}_2 = \underline{v}_{\mathrm{B}} \tag{5.36}$$

die nichtrelativistische vektorielle Addition von Geschwindigkeiten. Dieses Resultat ist also ebenfalls korrekt und dient somit der weiteren Plausibilisierung des untersuchten Gleichungssystems.

5.4 Solving the Equation System

Für die Berechnung der gesuchten Größen θ, χ und $\underline{n}_{\mathrm{B}}$ ist das System von Gleichungen aus Abschnitt 5.2 zu unhandlich, weswegen wir von den beiden Vektorgleichungen (5.21) und (5.22) einfachere skalare Gleichungen ableiten werden. Zur besseren Nachvollziehbarkeit notieren wir das Ausgangssystem (5.19) – (5.22) hier noch einmal:

$$\mathrm{ch}_1\,\mathrm{ch}_2 + \mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1, \underline{n}_2) \;=\; \cos\mathrm{ch} \tag{5.37}$$

$$\sin\mathrm{sh}\,(\underline{n}_{\mathrm{R}}, \underline{n}_{\mathrm{B}}) \;=\; 0 \tag{5.38}$$

$$\begin{aligned} \mathrm{sh}_1\,\mathrm{ch}_2\,\underline{n}_1 \;+\; \mathrm{sh}_2\,\mathrm{ch}_1\,\underline{n}_2 \;=\; \cos\mathrm{sh}\,\underline{n}_{\mathrm{B}} \;+\; \cdots \\ \cdots \;+\; \mathrm{sh}_1\,\mathrm{sh}_2\,\mathrm{th}\left[(\underline{n}_{\mathrm{B}}, \underline{n}_1)\,\underline{n}_2 \;-\; (\underline{n}_{\mathrm{B}}, \underline{n}_2)\,\underline{n}_1\right] \end{aligned} \tag{5.39}$$

$$\mathrm{sh}_1\,\mathrm{sh}_2\;\underline{n}_1 \times \underline{n}_2 \;=\; \sin\mathrm{ch}\,\underline{n}_{\mathrm{R}}. \tag{5.40}$$

Gleichung (5.39) entsteht aus (5.21) durch Ersetzen des Terms $\sin\underline{n}_{\mathrm{R}}$ durch (5.40) und anschließende Anwendung der $BAC - CAB$ Regel auf das so entstandene doppelte Kreuzprodukt. Aus diesem Gleichungssystem lassen sich die folgenden ergänzenden

Beziehungen ableiten:

$$\begin{aligned}
\mathrm{sh}_1\,\mathrm{ch}_2 \;+\; \mathrm{sh}_2\,\mathrm{ch}_1\,(\underline{n}_1,\underline{n}_2) \;&=\; \cos\mathrm{sh}\,(\underline{n}_{\mathrm{B}},\underline{n}_1) \;+\; \cdots \\
&\cdots + \mathrm{sh}_1\,\mathrm{sh}_2\,\mathrm{th}\,\big[(\underline{n}_{\mathrm{B}},\underline{n}_1)(\underline{n}_1,\underline{n}_2) \;-\; (\underline{n}_{\mathrm{B}},\underline{n}_2)\big] \qquad (5.41)\\
\mathrm{sh}_1\,\mathrm{ch}_2\,(\underline{n}_1,\underline{n}_2) \;+\; \mathrm{sh}_2\,\mathrm{ch}_1 \;&=\; \cos\mathrm{sh}\,(\underline{n}_{\mathrm{B}},\underline{n}_2) \;+\; \cdots \\
&\cdots + \mathrm{sh}_1\,\mathrm{sh}_2\,\mathrm{th}\,\big[(\underline{n}_{\mathrm{B}},\underline{n}_1) \;-\; (\underline{n}_{\mathrm{B}},\underline{n}_2)(\underline{n}_1,\underline{n}_2)\big] \qquad (5.42)\\
\mathrm{sh}_1\,\mathrm{sh}_2\,\sqrt{1-(\underline{n}_1,\underline{n}_2)^2} \;&=\; \sin\,\mathrm{ch}. \qquad (5.43)
\end{aligned}$$

Die Gleichungen (5.41) und (5.42) entstehen durch Bildung des Skalarproduktes von (5.39) mit den Richtungsvektoren $\underline{n}_1$ bzw. $\underline{n}_2$. Für (5.43) wurden die Beträge der in (5.40) auftretenden Vektoren mit Hilfe der $BAC - CAB$ Regel berechnet:

$$(\underline{n}_{\mathrm{R}},\underline{n}_{\mathrm{R}}) \;=\; 1 \qquad (5.44)$$

$$(\underline{n}_1\times\underline{n}_2)(\underline{n}_1\times\underline{n}_2) \;=\; \big(\underline{n}_1,\,\underline{n}_2\times(\underline{n}_1\times\underline{n}_2)\big) \;=\; 1-(\underline{n}_1,\underline{n}_2)^2. \qquad (5.45)$$

Das Quadrieren und anschließende Wurzelziehen erweckt den Eindruck eines möglichen Verlustes von Vorzeicheninformationen. Tatsächlich ist das nicht der Fall, da die Größen sh_1, sh_2 und sin ohne Beschränkung der Allgemeinheit positiv gewählt werden können:

- Die Faktoren $\mathrm{sh}_1 = \sinh(\chi_1/2)$ und $\mathrm{sh}_2 = \sinh(\chi_2/2)$ sind für negative Rapiditäten bzw. Geschwindigkeiten selbst negativ. Die von uns gewählte Parametrisierung der Lorentztransfromationen mit Hilfe der Richtungsvektoren $\underline{n}_1$ und $\underline{n}_2$ ermöglicht es uns, alle Informationen zur Orientierung der Geschwindigkeiten in den Richtungsvektoren auszudrücken. Die Rapiditäten χ_1 und χ_2 lassen sich auf diese Weise immer positiv wählen.

- Die Faktoren $\sin = \sin(\theta/2)$ und $\cos = \cos(\theta/2)$ sind aufgrund der verwendeten Halbwinkel im Bereich $0 \leq \theta \leq \pi$ beide nicht negativ. Jeder Winkel außerhalb dieses Bereiches kann durch Umkehr der Orientierung der Drehachse auf einen Winkel innerhalb dieses Bereiches zurückgeführt werden. Auch hier verhindert die geschickte Orientierung des Normalenvektors $\underline{n}_{\mathrm{R}}$ negative Koeffizienten.

Aus dem Gleichungssystem (5.37) – (5.43) können wir nun mit

dem Rotationswinkel θ der Thomas–Wigner Rotation
der Rapidität χ des resultierenden Lorentz–Boosts
dem Richtungsvektor $\underline{n}_{\mathrm{B}}$ der resultierenden Geschwindigkeit

alle gewünschten Größen berechnen.

Der Winkel der Thomas–Wigner Rotation:

Im ersten Schritt dividieren wir Gleichung (5.43) durch Gleichung (5.19) und eliminieren so den Term $\mathrm{ch} = \cosh(\chi/2)$. Wir erhalten einen Ausdruck für $\tan(\theta/2)$, aus dem mit Hilfe elementarer Umrechnungsformeln auch $\sin(\theta/2)$ und $\cos(\theta/2)$ bestimmt werden können:

$$\tan(\theta/2) = \frac{\mathrm{sh}_1\,\mathrm{sh}_2\,\sqrt{1-(\underline{n}_1,\underline{n}_2)^2}}{\mathrm{ch}_1\,\mathrm{ch}_2 + \mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1,\underline{n}_2)} \tag{5.46}$$

$$\sin(\theta/2) = \frac{\mathrm{sh}_1\,\mathrm{sh}_2\,\sqrt{1-(\underline{n}_1,\underline{n}_2)^2}}{\sqrt{\mathrm{ch}_1^2\,\mathrm{ch}_2^2 + \mathrm{sh}_1^2\,\mathrm{sh}_2^2 + 2\,\mathrm{ch}_1\,\mathrm{ch}_2\,\mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1,\underline{n}_2)}} \tag{5.47}$$

$$\cos(\theta/2) = \frac{\mathrm{ch}_1\,\mathrm{ch}_2 + \mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1,\underline{n}_2)}{\sqrt{\mathrm{ch}_1^2\,\mathrm{ch}_2^2 + \mathrm{sh}_1^2\,\mathrm{sh}_2^2 + 2\,\mathrm{ch}_1\,\mathrm{ch}_2\,\mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1,\underline{n}_2)}}. \tag{5.48}$$

Wie sich leicht nachvollziehen läßt, verschwindet die Thomas–Wigner Rotation für die im Abschnitt 5.3 betrachteten Sonderfälle:

$$\left.\begin{array}{ll} \text{Einzelne Boosts} & v_1 = 0,\ v_2 = 0: \\ \text{Parallele Boosts} & \underline{n}_1 = \pm\underline{n}_2: \\ \text{Galileischer Grenzfall} & v_1 \ll c,\ v_2 \ll c: \end{array}\right\} \theta = 0. \tag{5.49}$$

Rapidität des resultierenden Boosts:

Da nun die Ausdrücke $\sin(\theta/2)$ und $\cos(\theta/2)$ bekannt sind, kann sowohl Gleichung (5.37) als auch Gleichung (5.43) zur Berechnung der Rapidität χ herangezogen werden. Wir erhalten in beiden Fällen

$$\cosh(\chi/2) = \sqrt{\mathrm{ch}_1^2\,\mathrm{ch}_2^2 + \mathrm{sh}_1^2\,\mathrm{sh}_2^2 + 2\,\mathrm{ch}_1\,\mathrm{ch}_2\,\mathrm{sh}_1\,\mathrm{sh}_2\,(\underline{n}_1,\underline{n}_2)}\,. \tag{5.50}$$

Dieses Resultat reproduziert für die im letzten Abschnitt behandelten Sonderfälle die bekannten Ergebnisse:

$$\text{Einzelne Boosts} \quad v_1 = 0,\ v_2 = 0: \quad \chi = \chi_1,\ \ \chi = \chi_2 \tag{5.51}$$

$$\text{Parallele Boosts} \quad \underline{n}_1 = \pm\underline{n}_2: \quad \chi = \chi_1 \pm \chi_2 \tag{5.52}$$

$$\text{Galileischer Grenzfall} \quad v_1 \ll c,\ v_2 \ll c: \quad \underline{v} = \underline{v}_1 + \underline{v}_2. \tag{5.53}$$

Während sich die ersten beiden Beziehungen problemlos aus (5.50) ablesen lassen, ist zum Galileischen Grenzfall eine Erläuterung angebracht. Gleichung (5.53) entsteht aus

(5.50) durch Entwicklung aller Terme für kleine Rapiditäten $\chi \approx v/c$ bis zur zweiten Ordnung:

$$\begin{aligned}
1 + \frac{1}{8}\chi^2 &\approx \sqrt{1 + \frac{1}{4}\chi_1^2 + \frac{1}{4}\chi_2^2 + \frac{1}{2}\chi_1\chi_2\,(\underline{n}_1, \underline{n}_2)} \\
&\approx 1 + \frac{1}{8}\chi_1^2 + \frac{1}{8}\chi_2^2 + \frac{1}{4}\chi_1\chi_2\,(\underline{n}_1, \underline{n}_2) \\
\Rightarrow \quad \chi^2 &= \chi_1^2 + \chi_2^2 + 2\,\chi_1\chi_2\,(\underline{n}_1, \underline{n}_2) \qquad . \qquad (5.54) \\
\Rightarrow \quad v^2 &= v_1^2 + v_2^2 + 2\,v_1 v_2\,(\underline{n}_1, \underline{n}_2) \\
\Rightarrow \; (\underline{v}, \underline{v}) &= (\underline{v}_1, \underline{v}_1) + (\underline{v}_2, \underline{v}_2) + 2\,(\underline{v}_1, \underline{v}_2) = (\underline{v}_1 + \underline{v}_2,\, \underline{v}_1 + \underline{v}_2) \\
\Rightarrow \quad \underline{v} &= \underline{v}_1 + \underline{v}_2
\end{aligned}$$

Die Entwicklung des Ausdrucks (5.50) für kleine Geschwindigkeiten führt also erneut zum klassischen vektoriellen Additionsgesetz.

Richtungsvektor der resultierenden Geschwindigkeit:

Die zwei Komponenten des Richtungsvektors $\underline{n}_{\mathrm{B}}$ werden durch das System der beiden linearen Gleichungen (5.41) und (5.42) festgelegt:

$$\begin{aligned}
\mathrm{sh}_1\,\mathrm{ch}_2 + \mathrm{sh}_2\,\mathrm{ch}_1\,(\underline{n}_1, \underline{n}_2) &= \cos\mathrm{sh}\,(\underline{n}_{\mathrm{B}}, \underline{n}_1) + \cdots \\
&\cdots + \mathrm{sh}_1\,\mathrm{sh}_2\,\mathrm{th}\left[(\underline{n}_{\mathrm{B}}, \underline{n}_1)(\underline{n}_1, \underline{n}_2) - (\underline{n}_{\mathrm{B}}, \underline{n}_2)\right] \qquad (5.55) \\
\mathrm{sh}_1\,\mathrm{ch}_2\,(\underline{n}_1, \underline{n}_2) + \mathrm{sh}_2\,\mathrm{ch}_1 &= \cos\mathrm{sh}\,(\underline{n}_{\mathrm{B}}, \underline{n}_2) + \cdots \\
&\cdots + \mathrm{sh}_1\,\mathrm{sh}_2\,\mathrm{th}\left[(\underline{n}_{\mathrm{B}}, \underline{n}_1) - (\underline{n}_{\mathrm{B}}, \underline{n}_2)(\underline{n}_1, \underline{n}_2)\right]. \qquad (5.56)
\end{aligned}$$

Da der Vektor $\underline{n}_{\mathrm{B}}$ sich in der von $\underline{n}_1$ und $\underline{n}_2$ aufgespannten Ebene befindet, können wir letztere Vektoren als Basis verwenden und schreiben:

$$\underline{n}_{\mathrm{B}} = n^1\,\underline{n}_1 + n^2\,\underline{n}_2. \qquad (5.57)$$

Diese Basis ist im allgemeinen nicht orthogonal sondern schiefwinklig, und die im obenstehenden Gleichungssystem auftretenden kovarianten Komponenten $(\underline{n}_{\mathrm{B}}, \underline{n}_1)$ und $(\underline{n}_{\mathrm{B}}, \underline{n}_1)$ des Vektors $\underline{n}_{\mathrm{B}}$ müssen nicht mit den gesuchten kontravarianten Komponenten n^1 und n^2 übereinstimmen. Zwischen beiden Typen von Komponenten vermittelt der metrische Tensor $g_{ij} = (\underline{n}_i, \underline{n}_j)$, mit dessen Hilfe die Indizes herauf– bzw. heruntergezogen werden können:

$$n^i = g^{ij}\,(\underline{n}_{\mathrm{B}}, \underline{n}_j), \qquad (\underline{n}_{\mathrm{B}}, \underline{n}_i) = g_{ij}\,n^j. \qquad (5.58)$$

In unserem Fall haben der metrische Tensor g_{ij} und sein Inverses g^{ij} die Gestalt

$$(g_{ij}) = \begin{pmatrix} 1 & (\underline{n}_1, \underline{n}_2) \\ (\underline{n}_1, \underline{n}_2) & 1 \end{pmatrix} \tag{5.59}$$

$$(g^{ij}) = \begin{pmatrix} 1 & -(\underline{n}_1, \underline{n}_2) \\ -(\underline{n}_1, \underline{n}_2) & 1 \end{pmatrix} \frac{1}{1-(\underline{n}_1, \underline{n}_2)^2}, \tag{5.60}$$

womit sich die Beziehung zwischen kovarianten und kontravarianten Komponenten explizit hinschreiben läßt:

$$\begin{aligned} \big(1-(\underline{n}_1, \underline{n}_2)^2\big)\, n^1 &= (\underline{n}_{\mathrm{B}}, \underline{n}_1) + (\underline{n}_1, \underline{n}_2)(\underline{n}_{\mathrm{B}}, \underline{n}_2) \\ \big(1-(\underline{n}_1, \underline{n}_2)^2\big)\, n^2 &= (\underline{n}_{\mathrm{B}}, \underline{n}_2) + (\underline{n}_1, \underline{n}_2)(\underline{n}_{\mathrm{B}}, \underline{n}_1) \end{aligned} \tag{5.61}$$

$$\begin{aligned} (\underline{n}_{\mathrm{B}}, \underline{n}_1) &= n^1 + (\underline{n}_1, \underline{n}_2)\, n^2 \\ (\underline{n}_{\mathrm{B}}, \underline{n}_2) &= n^2 + (\underline{n}_1, \underline{n}_2)\, n^1 \end{aligned}. \tag{5.62}$$

Eine weitere hilfreiche Relation läßt sich aus (5.48) und (5.50) ableiten:

$$\mathrm{ch}_1\, \mathrm{ch}_2 + \mathrm{sh}_1\, \mathrm{sh}_2\, (\underline{n}_1, \underline{n}_2) = \cos(\theta/2) \cosh(\chi/2). \tag{5.63}$$

Mit Hilfe dieser Relation läßt sich später der Rotationswinkel θ aus dem Gleichungssystem (5.55) und (5.56) eliminieren. Wir beschreiben nun die notwendigen Umformungen des Gleichungssystems in einem kurzen Abriß:

- Jede der beiden Gleichungen wird zunächst mit dem Faktor $(\underline{n}_1, \underline{n}_2)$ multipliziert und anschließend von der anderen abgezogen.
- Die kovarianten Komponenten $(\underline{n}_{\mathrm{B}}, \underline{n}_j)$ werden mit Hilfe von (5.62) durch die kontravarianten Komponenten n^i ersetzt.
- In beiden Gleichungen entsteht vor jedem Term der Faktor $\big(1-(\underline{n}_1, \underline{n}_2)^2\big)$, den wir wegkürzen können. Als Folge dieser Division ist ist das Gleichungssystem nicht mehr für parallele Normalenvektoren $(\underline{n}_1, \underline{n}_2) = \pm 1$ gültig, da der obengenannte Faktor in diesem Fall verschwindet.

Nach diesen Umformungen hat das Gleichungssystem die Gestalt

$$\mathrm{ch}_2\, \mathrm{sh}_1 = \cos \mathrm{sh}\, n^1 - \mathrm{sh}_1\, \mathrm{sh}_2\, \mathrm{th}\big[n^2 + (\underline{n}_1, \underline{n}_2)\, n^1\big] \tag{5.64}$$

$$\mathrm{ch}_1\, \mathrm{sh}_2 = \cos \mathrm{sh}\, n^2 + \mathrm{sh}_1\, \mathrm{sh}_2\, \mathrm{th}\big[n^1 + (\underline{n}_1, \underline{n}_2)\, n^2\big]. \tag{5.65}$$

Jetzt ersetzen wir den Term cos durch Gleichung (5.63) und erhalten

$$\mathrm{ch}_2\, \mathrm{sh}_1 = \mathrm{ch}_1\, \mathrm{ch}_2\, \mathrm{th}\, n^1 - \mathrm{sh}_1\, \mathrm{sh}_2\, \mathrm{th}\, n^2 \tag{5.66}$$

$$\mathrm{ch}_1\, \mathrm{sh}_2 = \mathrm{sh}_1\, \mathrm{sh}_2\, \mathrm{th}\, n^1 + \big(\mathrm{ch}_1\, \mathrm{ch}_2 + 2\, \mathrm{sh}_1\, \mathrm{sh}_2\, (\underline{n}_1, \underline{n}_2)\big)\, \mathrm{th}\, n^2. \tag{5.67}$$

Dieses Gleichungssystem läßt sich folgendermaßen lösen:

$$\operatorname{th} n^1 = \frac{\operatorname{sh}_1}{\operatorname{ch}_1}\left(1 + \frac{\operatorname{sh}_2^2}{\operatorname{ch}^2}\right) \tag{5.68}$$

$$\operatorname{th} n^2 = \frac{\operatorname{ch}_2 \operatorname{sh}_2}{\operatorname{ch}^2}\,. \tag{5.69}$$

Wir verzichten an dieser Stelle auf die Überprüfung der Lösung durch Reduktion auf bekannte Spezialfälle und zeigen stattdessen im nächsten Abschnitt, daß die obenstehenden Komponenten n^1 und n^2 zur bekannten relativistischen Addition beliebig orientierter Geschwindigkeiten äquivalent sind.

5.5 Addition of Non Parallel Velocities

Gelegentlich ist in der Literatur und in Internetforen zu lesen, die Herleitung des relativistischen Additionstheorems zweier beliebig gerichteter Geschwindigkeiten sei etwas mühsam. Unser Ansatz (5.68) und (5.69) unterstreicht diese Aussage durchaus, da er dem bekannten Resultat (vgl. z.B. [5]) nicht im mindesten ähnelt. Der Grund des unterschiedlichen Erscheinungsbildes ist die Verwendung der Halbrapiditäten $\chi_1/2$, $\chi_2/2$ und $\chi/2$, die sich nicht direkt in Geschwindigkeiten ausdrücken lassen. Für die Umwandlung von Rapiditäten in Geschwindigkeiten mit Hilfe von

$$\tanh(\chi) = \frac{v}{c} \tag{5.70}$$

müssen wir daher hyperbolische Funktionen der vollen Rapidität generieren:

$$\cosh(\chi) = \cosh^2(\chi/2) + \sinh^2(\chi/2) \tag{5.71}$$

$$\sinh(\chi) = 2\cosh(\chi/2)\sinh(\chi/2) \tag{5.72}$$

$$\tanh(\chi) = \frac{2\tanh(\chi/2)}{1 + \tanh^2(\chi/2)}. \tag{5.73}$$

Für die Herleitung des Additionstheorems benötigten wir Terme, die zwischen Ausdrücken in Halbrapiditäten und Ausdrücken in vollen Rapiditäten vermitteln:

$$\tanh(\chi_i) = \frac{2\,\mathrm{ch}_i\,\mathrm{sh}_i}{\mathrm{ch}_i^2 + \mathrm{sh}_i^2} \qquad i \in \{1,2\} \tag{5.74}$$

$$1 + \tanh^2(\chi) = \frac{\mathrm{ch}^2 + \mathrm{sh}^2}{\mathrm{ch}^2} \tag{5.75}$$

$$\sqrt{1 - \tanh^2(\chi_1)} = \frac{1}{\mathrm{ch}_1^2 + \mathrm{sh}_1^2} \tag{5.76}$$

$$1 + \tanh(\chi_1)\tanh(\chi_2)\,(\underline{n}_1, \underline{n}_2) = \frac{\mathrm{ch}^2 + \mathrm{sh}^2}{\left(\mathrm{ch}_1^2 + \mathrm{sh}_1^2\right)\left(\mathrm{ch}_2^2 + \mathrm{sh}_2^2\right)}. \tag{5.77}$$

Diese Beziehungen lassen sich mit Hilfe der elementaren Rechenregeln für hyperbolische Funktionen sowie dem Ausdruck (5.50) für $(\underline{n}_1, \underline{n}_2)$ unschwer nachvollziehen. Die benötigten Geschwindigkeiten setzen wir aus Betrag und Richtungsvekor zusammen:

$$\frac{1}{c}\,\underline{v} = \tanh(\chi)\,\underline{n}, \qquad \frac{1}{c}\,\underline{v}_1 = \tanh(\chi_1)\,\underline{n}_1, \qquad \frac{1}{c}\,\underline{v}_2 = \tanh(\chi_2)\,\underline{n}_2. \tag{5.78}$$

Wir betrachten nun die Komponenten der resultierenden Geschwindigkeit

$$\frac{1}{c}\,\underline{v} = \tanh(\chi)\,n^1\,\underline{n}_1 + \tanh(\chi)\,n^2\,\underline{n}_2 \tag{5.79}$$

etwas genauer und eliminieren mit Hilfe von (5.74)–(5.77) alle Ausdrücke in Halbrapiditäten:

$$\begin{aligned}\tanh(\chi)\,n^1 &= 2\,\frac{\mathrm{ch}^2\,\mathrm{th}\,n^1}{\mathrm{ch}^2 + \mathrm{sh}^2} = 2\,\frac{\mathrm{sh}_1(\mathrm{ch}^2 + \mathrm{sh}_2^2)}{\mathrm{ch}_1(\mathrm{ch}^2 + \mathrm{sh}^2)} \\ &= \frac{\tanh(\chi_1) + \tanh(\chi_2)\left(1 - \sqrt{1 - \tanh^2(\chi_1)}\right)(\underline{n}_1, \underline{n}_2)}{1 + \tanh(\chi_1)\tanh(\chi_2)\,(\underline{n}_1, \underline{n}_2)}\end{aligned} \tag{5.80}$$

$$\begin{aligned}\tanh(\chi)\,n^2 &= 2\,\frac{\mathrm{ch}^2\,\mathrm{th}\,n^2}{\mathrm{ch}^2 + \mathrm{sh}^2} = 2\,\frac{\mathrm{ch}_2\,\mathrm{sh}_2}{\mathrm{ch}^2 + \mathrm{sh}^2} \\ &= \frac{\tanh(\chi_2)\sqrt{1 - \tanh^2(\chi_1)}}{1 + \tanh(\chi_1)\tanh(\chi_2)\,(\underline{n}_1, \underline{n}_2)}.\end{aligned} \tag{5.81}$$

Die obenstehenden Beziehungen können am einfachsten in umgekehrter Schlußrichtung überprüft werden, indem man jeweils die Terme mit vollen Rapiditäten auf den vorgegebenen Ausdruck in Halbrapiditäten zurückrechnet. Schließlich fassen wir die beiden Komponenten (5.80) und (5.81) gemäß (5.79) in einer Vektorgleichung zusammen, ersetzen die Rapiditäten durch Geschwindigkeiten und erhalten

$$\frac{1}{c}\,\underline{v} \;=\; \frac{\frac{1}{c}\,\underline{v}_1 + \frac{1}{c}\,v_2\,(\underline{n}_1,\underline{n}_2)\,\underline{n}_1 + \sqrt{1-v_1^2/c^2}\;\frac{1}{c}\,v_2\big[\underline{n}_2 - (\underline{n}_1,\underline{n}_2)\,\underline{n}_1\big]}{1+(\underline{v}_1,\underline{v}_2)/c^2}. \tag{5.82}$$

Offenbar tragen die zu $\underline{v}_1$ parallel bzw. senkrecht stehenden Anteile von $\underline{v}_2$ in verschiedener Weise zur resultierenden Geschwindigkeit bei:

$$\underline{v}_{2\parallel} = v_2\,(\underline{n}_1,\underline{n}_2)\,\underline{n}_1 \qquad \text{(paralleler Anteil)} \tag{5.83}$$

$$\underline{v}_{2\perp} = v_2\big[\underline{n}_2 - (\underline{n}_1,\underline{n}_2)\,\underline{n}_1\big] \qquad \text{(senkrechter Anteil).} \tag{5.84}$$

Mit den Abkürzungen $\underline{v}_{2\parallel}$ und $\underline{v}_{2\perp}$ lautet das relativistische Additionstheorem für beliebig gerichtete Geschwindigkeiten

$$\underline{v} \;=\; \frac{1}{1+(\underline{v}_1,\underline{v}_2)/c^2}\left[\underline{v}_1 + \underline{v}_{2\parallel} + \sqrt{1-v_1^2/c^2}\;\underline{v}_{2\perp}\right]. \tag{5.85}$$

Diese Form des Additionstheorems findet man auch in der Standardliteratur [5], wo es auf weit einfachere Weise hergeleitet wird als hier. Damit ist Gleichung (5.85) eine weitere Bestätigung des Gleichungssystems (5.19) – (5.22) sowie der zugrundeliegenden Darstellung (4.25) von Lorentz–Boosts als hermitesche Matrizen aus der Gruppe SL(2,$\mathbb{C}$). Darüberhinaus ist uns in diesem Abschnitt gelungen, das Additionstheorem für beliebig orientierte Geschwindigkeiten ausgehend von allgemeinen algebraischen Eigenschaften der Lorentztransformation in einem Top– Down– Verfahren abzuleiten.

5.6 Discussion of the Rotation Angle

Um ein Gefühl für die Eigenschaften des Rotationswinkels θ zu bekommen, verwenden wir Gleichung (5.46) und schreiben sie folgendermaßen um:

$$\tan(\theta/2) \;=\; \frac{\mathrm{th}_1\,\mathrm{th}_2\,\sqrt{1-(\underline{n}_1,\underline{n}_2)^2}}{1+\mathrm{th}_1\,\mathrm{th}_2\,(\underline{n}_1,\underline{n}_2)} \;=\; \frac{\mathrm{th}_1\,\mathrm{th}_2\,\sin(\gamma)}{1+\mathrm{th}_1\,\mathrm{th}_2\,\cos(\gamma)}. \tag{5.86}$$

Das Skalarprodukt $(\underline{n}_1,\underline{n}_2)$ haben wir im letzten Schritt durch den Winkel γ zwischen den Geschwindigkeitsvektoren der beiden Lorentz–Boosts ausgedrückt. Die Größen th_1 und th_2 hatten wir als Abkürzung für $\tanh(\chi_1/2)$ und $\tanh(\chi_2/2)$ eingeführt. Es sind monotone Funktionen der Rapiditäten und somit auch der Geschwindigkeiten v_1 bzw. v_2 mit den Randwerten

$$\left.\begin{array}{llll} \mathrm{th}_i & = 0 & \text{für} & v_i = 0 \\ \mathrm{th}_i & = 1 & \text{für} & v_i = c \end{array}\right\}\; i \in \{1,2\}. \tag{5.87}$$

Gemäß Gleichung (5.86) hängt die Funktion $\tan(\theta/2)$ ebenfalls monoton von den beiden Variablen th_1 und th_2 ab, denn die partiellen Ableitungen

$$\frac{\partial \tan(\theta/2)}{\partial\,(\mathrm{th}_1)} = \frac{\partial \tan(\theta/2)}{\partial\,(\mathrm{th}_2)} = \frac{1}{\big(1+\mathrm{th}_1\,\mathrm{th}_2\,\cos(\gamma)\big)^2} \tag{5.88}$$

sind überall positiv. Den größten Wert $\theta_{\scriptscriptstyle M}$ erreicht der Rotationswinkel wenn beide Geschwindigkeiten sich mit $v_1 \to c$ und $v_2 \to c$ dem Wert der Lichtgeschwindigkeit nähern:

$$\tan(\theta_{\scriptscriptstyle M}/2) = \frac{\sin(\gamma)}{1+\cos(\gamma)} = \frac{2\cos(\gamma/2)\sin(\gamma/2)}{1+\cos^2(\gamma/2)-\sin^2(\gamma/2)} = \tan(\gamma/2). \tag{5.89}$$

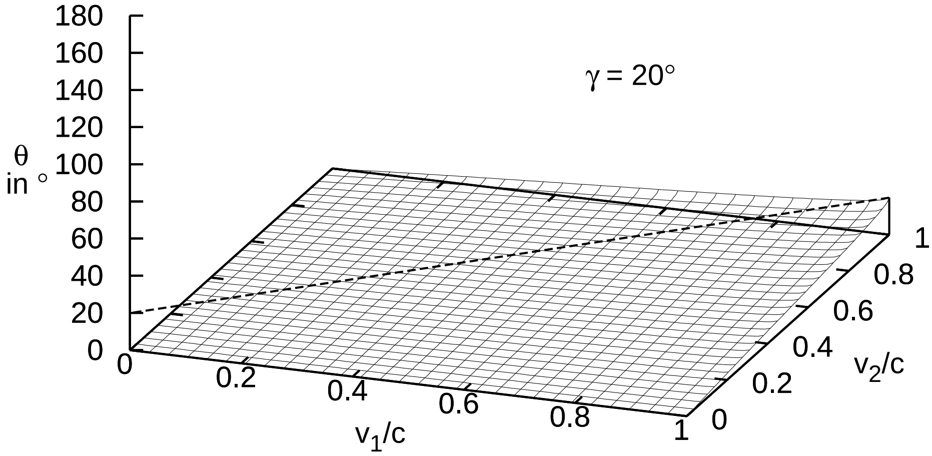
Hieraus können wir schließen daß der maximale Winkel

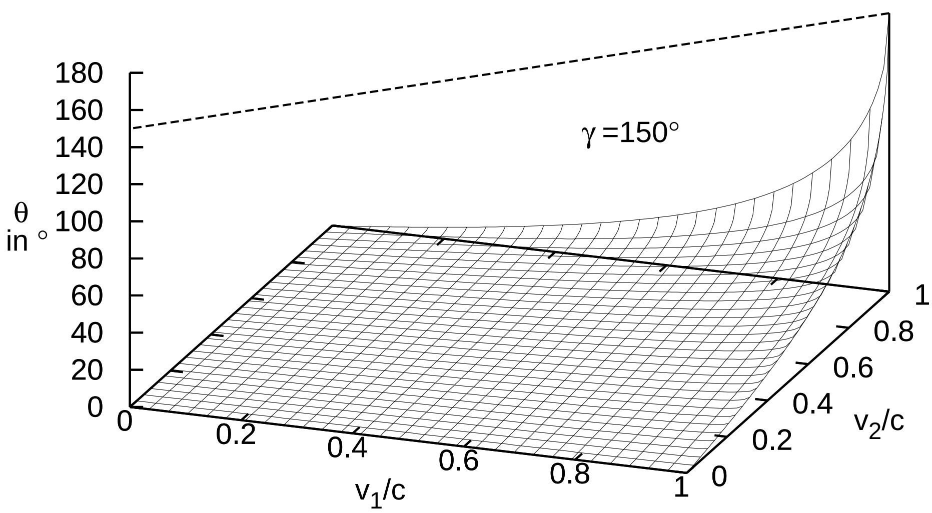
$$\theta_{\scriptscriptstyle M} = \gamma \tag{5.90}$$

lautet. In Abbildung 5.1 ist der Rotationswinkel θ in Abhängigkeit der beiden Geschwindigkeiten v_1 und v_2 für $\gamma = 20°$ und $\gamma = 150°$ dargestellt. Letztendlich können wir die Eigenschaften der Thomas–Wigner Rotation folgendermaßen in Worte fassen:

> Zwei aufeinanderfolgende Lorentz–Boosts lassen sich durch einen einzigen Boost beschreiben, wenn das System zusätzlich innerhalb der von den Geschwindigkeiten $\underline{v}_1$ und $\underline{v}_2$ aufgespannten Ebene um den Winkel θ gedreht wird. Diese Drehung verlegt eine Gerade entlang $\underline{v}_1$ so, daß sie anschließend zwischen den Vektoren $\underline{v}_1$ und $\underline{v}_2$ verläuft. Falls beide Geschwindigkeiten die Größenordnung der Lichtgeschwindigkeit erreichen, verläuft diese Gerade nach der Drehung entlang $\underline{v}_2$.

Figure 5.1: Winkel der Thomas–Wigner–Rotation





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